Dictator Problems

written by nathan hart-hodgson on Functor Network
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Welcome to my first blog post! As your Resident Assistant and a fellow math enthusiast, I'm excited to kick off with some classics and variations of what I call "Dictator Problems" - problems where a group of mathematicians must devise strategies to outsmart a dictator's twisted challenges.

These problems are not just fun brain teasers—they also delve into fascinating concepts in combinatorics, probability, and infinite sets. Each problem is an opportunity to explore deep mathematical ideas in a playful and engaging way. So dive in, give them a try, and see how many of these mathematicians you can help escape!

Without further ado, here are the Dictator Problems:

1. A dictator wants to send all his 100 mathematicians into prison (because he thinks they are responsible for all the troubles in the country). Still, he gives them a chance to escape punishment. The 100 mathematicians will be randomly arranged into a queue, standing one behind the other, and a guard will place a blue or red cap on their heads. Each mathematician can see the colors of all caps in front of them, but not their own, nor those behind them. Starting from the end of the queue, each mathematician must say either "red" or "blue." Those who correctly guess the color of their cap will be released. The mathematicians may agree on a strategy in advance. How many of them can escape?

2. The same dictator wants to imprison the previously escaped 99 mathematicians again. This time, the rules are different. Again, a guard will place a blue or red cap on their heads, but now each mathematician can see the color of every cap except their own. They must guess simultaneously, without knowing the guesses of the others. They will escape only if all of them guess correctly, but each mathematician can provide two guesses: guess A and guess B. They must write down both guesses for the color of their own cap, and if all A guesses are correct or all B guesses are correct, then they all escape; otherwise, they are all executed. Can they surely escape if they agree on a strategy in advance?

3. In an even worse and much larger dictatorship, there are infinitely many mathematicians in prison. Each mathematician gets one last chance to survive. A guard will place a blue or red cap on their heads, so each mathematician can see the color of all caps except their own. At once, everyone must write down their guess for the color of their own cap. If

only finitely many of them guess incorrectly, then all will be released; otherwise, all of them will be executed. The mathematicians may agree on a strategy in advance. Show that there exists a strategy that guarantees their survival.

Bonus: What strategy will ensure the least number of errors?

4. Two previously mentioned dictators have met recently, discussed their experiences with imprisoning mathematicians, and decided to combine their challenges. They imprison infinitely many mathematicians again. As before, a guard places a blue or red cap on each mathematician's head, and each can see the color of all caps except their own. They must guess simultaneously, without knowing the guesses of the others. This time, they will escape if all of them guess correctly, but again, each mathematician can provide two guesses: guess A and guess B. If all A guesses are correct or all B guesses are correct, they all escape. Show that there exists a strategy that guarantees their escape.

Remember, every attempt at solving these problems could earn you a cool prize—so give it your best shot!

-Nathan Hart-Hodgson

Hints

Below are the hints. Use them to guide you through the problems, but remember to challenge yourself before peeking!

1. Use Parity
2. u can use consistent guessing strategies to ensure all mathematicians are correct for one of their guesses
3. Equivalence relation $\{x\} \sim \{y\}$ if they differ in finitely many places.
4. Try combining previous strategies.