# Lagrangian Framework for Fluids Suggest Symmetry-Breaking Mechanism for Turbulent Flows

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### Brief Note on the Development of the Framework

Fluid dynamics often relies on the Navier-Stokes equations (NSE) to model various flow phenomena. I have here a theoretical framework which recasts the equations of fluid mechanics in a Lagrangian formalism in a way where it can be shown that the kinetic energy of the vorticity explicitly break the scaling symmetry of the theory. This explicit symmetry breaking potentially provides a mechanism by which energy cascades and dissipation occur in turbulent flows. Additionally it demonstrates how the inclusion of vorticity fixes a scale.

This framework was developed from abstract symmetry principles and aims to address multiple problems in physics. My first goal was to understand how motion couples to the underlying geometry resulting in emergent dynamics. Fluid dynamics are fundamentally "shaped" by geometry, so I began by exploring simple fluid systems.

Some brief calculations already suggest the role of local scaling symmetries. If we define a conservative vector field to model a wave propagating in a pipe with cross sectional area $S$

$$v = -\nabla\phi$$

we know that it is by definition irrotational: $\nabla \times v = 0$. The mass flow field $j = \rho S v$ is also irrotational for constant cross sectional areas $\nabla \times j = 0$. Thus the field $\phi$ exhibits self-similarity at a scale (for any choice of $S$). However if we locally fix a scale by promoting $S \to S(x)$ to a varying cross sectional area, we see that the mass-flow becomes rotational

$$\nabla \times j = \rho \nabla S \times \nabla\phi \neq 0.$$

This is a very basic derivation but I hope the results are clear, locally fixing a scale introduces additional dynamics, particularly rotational dynamics.

In a similar manner, by demanding that a real scalar field $\phi$ be invariant under local scaling transformations a rotational vector potential $V^\mu$ must be introduced resulting in a Helmholtz-decomposition. An interesting consequence is that the symmetry of the Lagrangian is explicitly broken by the kinetic term for the vector potential, again suggesting the association between vorticity and scale-fixing.

Another result is that the homogeneous equations for the vector potential imply that the field is irrotational and thus conservative.

## Theoretical Background

**Scaling Symmetry and CFT** Delving deeper into the symmetry

$$\phi \to \sigma\phi, \tag{1}$$

where $\sigma$ is the scaling factor, ties to conformal field theory (CFT) are considered. Global scaling symmetry is almost universal as it corresponds to the trivial case where $\sigma$ is a constant, the more relevant and interesting cases are the local scaling symmetry that occurs when $\sigma(x^\mu)$ becomes a function of spacetime coordinates $x^\mu$.

We recognize that the scaling transformation $\sigma(x^\mu)$ is a conformal dilation, indicating that the group of scalar matrices $\mathcal{C}$ is a subset of the conformal group $\mathcal{G}$:

$$\mathcal{C} \subset \mathcal{G}.$$

This conformal dilation is a Weyl transformation, defined as the local scaling of the metric tensor:

$$g_{ab} \to \Omega(x^\mu)^{-2} g_{ab},$$

where the conformal factor $\Omega(x^\mu) = e^{\omega(x^\mu)}$ generates a new metric $\tilde{g}_{ab} = \Omega(x^\mu)^{-2} g_{ab}$ in the same conformal class as $g_{ab}$. The field $\phi$ has a conformal weight $\Delta$, where $\sigma = e^{\Delta\alpha(x^\mu)}$. Identifying $\sigma$ with $\Omega$, we can conclude that $\alpha = \omega$. Combining Equation (1) with

$$g_{\mu\nu} \to \sigma^{-2} g_{\mu\nu} \tag{2}$$

we have two transformation laws. In order to understand how $V^\mu$ transform, we need to formally define the theory.

## Formulation of Scalar Field Theory

To ensure that the symmetry is respected, the covariant derivative

$$D^\mu = \partial^\mu - V^\mu$$

is introduced, where the field $V^\mu = (\mathbf{p}, \mathbf{v})$ has components of pressure and velocity. In order for the derivative of $\phi$ to respect the symmetry we need $D^\mu\phi$ to transform the same as $\phi$ so

$$(D^\mu\phi) \to \sigma(D^\mu\phi) \tag{3}$$

or equivalently $D^\mu(\sigma\phi) = \sigma(D^\mu\phi)$. To satisfy Equation (3), we conclude that $V^\mu$ must transform as

$$V^\mu \to V^\mu + \sigma^{-1}\partial^\mu\sigma \tag{4}$$

where $\delta V^\mu = \sigma^{-1}\partial^\mu \sigma$. Now we have the full set of transformations, Equations (1), (2), (3) and (4)

$$\phi \to \sigma\phi$$
$$g_{\mu\nu} \to \sigma^{-2} g_{\mu\nu}$$
$$(D^\mu \phi) \to \sigma(D^\mu \phi)$$
$$V^\mu \to V^\mu + \sigma^{-1}\partial^\mu \sigma.$$

We can then modify our Lagrangian for a free scalar field $\mathcal{L}(\phi, \partial_\mu \phi) \to \mathcal{L}(\phi, D_\mu \phi)$ so the minimal coupling for this theory is

$$\mathcal{L} = (D_\mu \phi)(D^\mu \phi). \tag{5}$$

Equation (5) has the equations of motion (EOM)

$$\tilde{\Box}\phi = 0 \tag{6}$$

where $\tilde{\Box} = D_\mu D^\mu$. Expanding the terms we get

$$\partial_\mu \partial^\mu \phi - V_\mu \partial^\mu \phi - V^\mu \partial_\mu \phi - \partial_\mu V^\mu \phi + V^\mu V_\mu \phi = 0.$$

Note that both Equations (5) and (6) obey the conformal symmetry we have outlined.

**Explicit Symmetry Breaking**

Equation (5) describes the minimal coupling but for a complete description of the system we need a kinetic term for $V^\mu$. I defined the term

$$\mathcal{L}_V = -\frac{1}{4}\Omega_{\mu\nu}\Omega^{\mu\nu}$$

where $\Omega^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$ as is typical for vector fields in gauge theory. Note that $\Omega^{\mu\nu}$ is the vorticity tensor, and that the term $\mathcal{L}_V$ is not symmetric under the conformal transformations

$$-\frac{1}{4} g_{\mu\alpha} g_{\beta\nu} \Omega^{\mu\nu}\Omega^{\alpha\beta} \to -\frac{1}{4}\sigma^{-4} g_{\mu\alpha} g_{\beta\nu} \Omega^{\mu\nu}\Omega^{\alpha\beta}. \tag{7}$$

This means that including the kinetic term for the field $V^\mu$ results in the explicit breaking of the conformal symmetry and thus fixes a scale. The total Lagrangian

$$\mathcal{L}_T = (D_\mu \phi)(D^\mu \phi) - \frac{1}{4}\Omega_{\mu\nu}\Omega^{\mu\nu} \tag{8}$$

can now be used to describe the whole system. The EOM when varying with respect to $V^\mu$ are

$$\partial_\mu \Omega^{\mu\nu} = j^\nu \tag{9}$$

which can be combined with Equation (6) to obtain

$$\tilde{\Box}\phi = 0$$
$$\partial_\mu \Omega^{\mu\nu} = j^\nu.$$

Here Equation (6) respects the symmetry whereas Equation (9) does not suggesting that the scalar potential field maintains its invariance.

Some calculations reveal that the current can be defined as

$$j^\mu = -4(D^\mu \phi)\phi = -2\pi^\mu \phi \tag{10}$$

where $\pi^\mu$ is the canonical momentum. Expanding

$$j^\mu = -4(\partial^\mu \phi)\phi + 4V^\mu \phi^2$$

it becomes clear that $\phi$ couples quadratically to the rotational field. Further investigations suggests the current to be associated with mass-flow rates, but first we need to use the framework outlined to actually represent fluid systems before we can further analyze.

### Fluid Representation

It may not seem immediately obvious that Equation (8) can represent a fluid system, indeed it took me a while to come to that conclusion. Especially in the case of the field $V^\mu$ which I struggled to interpret physically for quite some time. I have landed on a Maxwell-like representation that I am satisfied with, and so far the results have been promising. Indeed the theory already greatly resembles classical electrodynamics so why not continue with the analogy?

The scalar potential is easy to represent. The state of a fluid can be given by it's pressure and velocity $p$ and $v$ respectively so we define the four-vector

$$U^\mu = -\partial^\mu \phi \tag{11}$$

which has components $U^\mu = (\sqrt{\beta}p, \sqrt{\rho_0}v)$ where $\frac{1}{\sqrt{\beta\rho_0}}$ is the speed of sound. For the time being we will work dimensionless to simplify the calculations, the constants will be relevant later. Combined with the incompressible field equations derived from the NSE we have

$$\frac{\partial v}{\partial t} = -\nabla p$$
$$\nabla \times v = 0$$

thus $\phi$ adequately describes incompressible flow consistent with then NSE. Note that the EOM of this theory, $\mathcal{L} = U_\mu U^\mu$, are that of a free scalar field

$$\partial_\mu U^\mu = 0 \tag{12}$$

which will become relevant later for gauge fixings.

For the field $V^\mu$ I used a Maxwell-like approach by defining two vector fields in 3 dimensional space $U$ and $W$ which I have denoted the transport (or convective acceleration) and vorticity respectively. Similar to electric and magnetic fields

$$U = (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \mathbf{p} - \frac{\partial \mathbf{v}}{\partial \mathbf{t}} \tag{13}$$

$$W = \nabla \times \mathbf{v} \tag{14}$$

Where $U$ is analogous to the electric field and $W$ the magnetic field. These fields allow us to represent and analyze rotational flow in ways that are somewhat familiar. Using Equations (13), and (14) we can reinterpret Equation (9) as a set of Maxwell-like equations and their integral forms

$$\nabla \cdot U = \frac{\rho}{\rho_0}, \oint U \cdot dS = \frac{Q}{\rho_0} \tag{15}$$

$$\nabla \cdot W = 0, \oint W \cdot dS = 0 \tag{16}$$

$$\nabla \times U = -\frac{\partial W}{\partial t}, \oint U \cdot dl = -\frac{\partial \Gamma}{\partial t} \tag{17}$$

$$\nabla \times W = \frac{\partial U}{\partial t}, \oint W \cdot dl = \frac{\partial Q}{\partial t}. \tag{18}$$

Here the intricate interactions contained in Equation (9) can be appreciated with this representation, where they manifest as exchanges between $U$ and $W$.

Additionally it follows from Maxwell theory that there exists stable self-propagating waves which I am, creatively, calling UW waves. Taking the curl of Equations (17) and (18) we get

$$\Box U = -\nabla \rho \tag{19}$$

$$\Box W = 0. \tag{20}$$

I think that UW waves could be useful for describing large stable structures in turbulent flows, however more validation is needed. Vortex shedding is a particular physical case I can think of where there are stable self-propagating vorticity waves $W$. The shedding of vortices in alternating directions reflects the Maxwell-like nature of this system. It is also relevant that rotation in the system explicitly breaks the symmetry of Equation (8) which fixes a scale. This is particularly relevant as it aligns with known concepts in the study of turbulent flows.