Volume-constrained Weyl Surfaces

Eowyn • 8 Mar 2026

Introduction

If you are not familiar with my work, I am interested in geometric fluids: the idea that the geometry of a fluid configuration can encode dynamical information. For example, consider a cubic glass of water and a cylindrical glass containing the same volume. It is clear, both intuitively and mathematically, that perturbations of the fluid surface will lead to different dynamics because the geometry of the container differs.

Additionally, fluids select preferred space-time scales determined by physical parameters such as volume, density, and intermolecular forces.

One way to study the interaction between scaling and dynamics is through Weyl geometry. In this approach the background geometry, with metric $g_{\mu\nu}$ in $3+1$ space-time dimensions, plays the role of the container while the fluid degrees of freedom live on top of this geometry.

The following example began as a simple toy model, but it illustrates several interesting features of how Weyl-type scaling transformations can appear in fluid systems.

Conformal Surface Fluctuations

Consider a glass containing some liquid. For simplicity we work in two spatial dimensions so the glass has horizontal length l and the fluid inside has height h.

Let

$$\phi(x) = h, \qquad 0 \leq x \leq l$$

be a scale function describing the level of the liquid.

We now gently perturb the container so that the surface of the fluid fluctuates without creating any discontinuities. The scale field therefore changes according to

$$\phi \to \tilde{\phi}. \tag{1}$$

The main physical assumption is volume conservation,

$$A_0 = A \tag{2}$$

where the initial area is

$$A_0 = hl$$

and the perturbed area is

$$A = \int_0^l \tilde{\phi}(x)\, dx.$$

We now interpret the transformation (1) as a Weyl rescaling of the scale field,

$$\tilde{\phi}(x) = e^{\omega(x)}\phi(x) = h\, e^{\omega(x)}.$$

Substituting into the area functional gives

$$A = h \int_0^l e^{\omega(x)} dx.$$

Imposing the conservation condition (2) yields

$$\frac{1}{l} \int_0^l e^{\omega(x)} dx = 1.$$

It is convenient to write this in terms of a spatial average

$$\langle e^{\omega} \rangle = 1, \tag{3}$$

where

$$\langle f \rangle = \frac{1}{l} \int_0^l f(x)\, dx.$$

Equation (3) is therefore simply a restatement of the volume conservation constraint.

To study small surface fluctuations we take

$$|\omega| \ll 1.$$

Expanding the exponential,

$$e^{\omega} = 1 + \omega + \frac{1}{2}\omega^2 + \mathcal{O}(\omega^3).$$

Taking the spatial average gives

$$\langle e^{\omega} \rangle = 1 + \langle \omega \rangle + \frac{1}{2}\langle \omega^2 \rangle + \cdots .$$

Enforcing the constraint (3) therefore requires

$$\langle \omega \rangle = 0,$$

thus the fluctuation field must have zero average over the domain.

Weyl-rescaling

As a simple example we choose a standing wave fluctuation

$$\omega(x) = \varepsilon \sin(kx)$$

where

$$k = \frac{2\pi n}{l}, \qquad \varepsilon \ll 1.$$

Although this function satisfies

$$\langle \omega \rangle = 0,$$

the nonlinear constraint

$$\langle e^{\omega} \rangle = 1$$

is not automatically satisfied because higher powers of ω contribute to the exponential. To restore the constraint we introduce a constant shift

$$\tilde{\omega}(x) = \omega(x) + c.$$

The area becomes

$$A = h \int_0^l e^{\tilde{\omega}(x)} dx = h e^c \int_0^l e^{\omega(x)} dx.$$

Imposing $A = A_0$ gives

$$e^c \frac{1}{l} \int_0^l e^{\omega(x)} dx = 1.$$

Therefore the constant shift is

$$c = -\ln\left(\frac{1}{l} \int_0^l e^{\omega(x)} dx \right).$$

This global rescaling ensures that the volume constraint is always satisfied. Geometrically, the fluctuating field $\omega(x)$ generates local conformal rescalings of the surface height, while the constant mode c acts as a global scale adjustment that enforces the volume constraint. In this sense the model illustrates how local Weyl-type transformations can interact with global conservation laws in a fluid system.

Here is a Desmos model demonstrating how the local surface oscillations cannot occur independently. When the surface develops peaks and troughs, the exponential factor increases the total area slightly. The constant shift c therefore rescales the entire surface downward so that the total volume remains fixed. In a standing wave visualization this appears as a slight tilting or "swiveling" of the nodes as the system compensates for the nonlinear area change.

Reflection

This toy model illustrates a simple but interesting interaction between conformal scaling and fluid constraints. By allowing the surface height to fluctuate through a local Weyl-type transformation

$$\phi(x) \to e^{\omega(x)}\phi(x),$$

the geometry of the fluid surface acquires a fluctuating conformal factor. However, the physical requirement of volume conservation imposes the global constraint

$$\langle e^{\omega} \rangle = 1.$$

Because the exponential is nonlinear, local fluctuations generally increase the total area. To compensate for this effect we introduce a constant shift of the fluctuation field,

$$\tilde{\omega}(x) = \omega(x) + c,$$

which acts as a global rescaling of the surface. The constant c is determined uniquely by the volume constraint and effectively fixes the zero mode of the fluctuation field.

From a geometric perspective this produces an interesting interplay between local and global behavior. The field $\omega(x)$ generates local conformal deformations of the surface, while the constant mode c enforces a global conservation law. As a result, oscillatory modes of the surface are not completely independent: changes in the local geometry must be compensated by a global rescaling to preserve the total volume.

Although the model is extremely simple, it captures an important structural idea that appears in more sophisticated geometric theories. In particular, it demonstrates how Weyl-type scaling transformations can interact with conservation constraints in continuum systems. In the context of geometric fluid models, this suggests that scale fields may play a natural role in describing surface fluctuations and their coupling to global fluid properties.

While the present construction is only a toy model, it provides a useful visualization of how conformal geometry and fluid constraints can work together, and it may serve as a starting point for exploring more realistic Weyl-geometric descriptions of fluid dynamics. I thought this was very interesting! I have essentially only been studying conformal geometries in fluid systems, but I know they are relabent in gauge gravities and string theory, so please let me know if you notice anything!