Vortices and Topological Defects

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Lamb-Oseen and Point Vortices

In the context of the gauge-invariant fluid theory derived from a (pseudo) Weyl-symmetric scalar field discussed in my last post, vortices emerge as localized structures in the solenoidal gauge field V^μ, whose field strength $\Omega^{\mu\nu}$ plays the role of a vorticity tensor. Two representative vortex configurations serve as fundamental examples for analysis: the idealized point vortex and the physically regularized Lamb-Oseen vortex.

The point vortex, defined as $\mathbf{v}(r) = \frac{\Gamma}{2\pi r}\hat{\theta}$, exhibits a singular core and topological winding in the phase of the gauge potential

$$\frac{1}{2\pi}\oint_{\mathcal{C}} \nabla\theta \cdot dl = 1$$

Despite its singularity, the circulation Γ is finite and conserved, due to the decay of the velocity field as $\frac{1}{r}$. The vorticity is given by $W = \Gamma\delta^2(\mathbf{x})$. In contrast, the Lamb-Oseen vortex provides a smooth, finite-core alternative, governed by

$$\mathbf{v}(r) = \frac{\Gamma}{2\pi r}(1 - e^{-\frac{r^2}{r_0^2}})\hat{\theta},$$

where r_0 sets the vortex core scale.

$$W(r) = \frac{1}{2\pi r_0^2} e^{-\frac{r^2}{r_0^2}}$$

This regularization enables direct computation of quantities such as the emergent geometric potential $\mathcal{V}_{geo} = |\mathbf{v}|^2$ and the location of the scalar field.

Circulation, Vorticity, and Their Invariance

Circulation in this model is defined analogously to classical fluid dynamics:

$$\Gamma = \oint_{\mathcal{C}} \mathbf{v} \cdot dl$$

where $\mathcal{C}$ is a loop enclosing the vortex. Due to the solenoidal nature of $\mathbf{v}$ and the structure of the theory, Γ is a gauge-invariant quantity. Importantly, Γ is also topologically protected: its quantization or conservation is tied to the winding number of the gauge field around singularities or regions of high curvature. This emphasizes vortices as a local phenomenon since constant curvatures result in varying circulation.

The vorticity field $W = \nabla \times \mathbf{v}$, derived from the spatial components of $\Omega^{\mu\nu}$, is another gauge-invariant observable. For a point vortex, W is singular and concentrated at a point, while for a Lamb-Oseen vortex, W is a smooth Gaussian function centered at the origin. In both cases, $\nabla \cdot W = 0$, indicating that vorticity behaves like a conserved flux density, analogous to a magnetic field in electromagnetism.

Emergence of Nontrivial Topology via the Gauge Field

The presence of nonzero circulation and localized vorticity corresponds to nontrivial topology in the gauge field configuration. Because the gauge group $G = C^{\infty}(\mathcal{M}, \mathbb{R}^{\times})$ is disconnected, with $\pi_0(G) = \mathbb{Z}_2$, the gauge field V^{μ} can host configurations that are topologically distinct and not connected via smooth gauge transformations. This allows for the existence of topological defects—vortices whose core structure and surrounding field geometry cannot be removed by gauge fixing.

In particular, vortex configurations are characterized by a nontrivial holonomy: the integral of V^{μ} around a loop encodes global information about the gauge sector, effectively labeling distinct topological classes. These classes persist under evolution and deformation, provided singularities do not annihilate or merge, thereby giving rise to robust structures in the theory. The coupling of the scalar field ϕ to the gauge field leads to further structure: ϕ becomes localized in the geometric potential $\mathcal{V}_{geo}$, which in turn reflects the topological character of V^{μ}.

Thus, in this gauge-invariant fluid model, vortices are not merely dynamic features—they are manifestations of the underlying topological structure enforced by the gauge symmetry and the geometry of the scalar field's interaction.