Proving absence, numerosity and existence

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Proof of Absence

There are no distinct natural numbers x and y that make $\frac{x^3+y^3}{1/x+1/y}$ a square number.

1

If u and v are coprime, then u and $u+v$ are coprime.

2

For two odd numbers $2m-1$ and $2n-1$, $(2m-1)^2+(2n-1)^2$ is even but not divided by 4 while $(2m-1)^2-(2n-1)^2$ is divided by 4.

3

Sum of two coprime square numbers is not divided by 3 because $(3m-1)^2+(3n-1)^2$, $(3m-1)^2+(3n)^2$, $(3m-1)^2+(3n+1)^2$ and $(3m)^2+(3n+1)^2$ are not divided by 3.

4

Dividing $\frac{x^3+y^3}{1/x+1/y} = xy(x^2-xy+y^2) = z^2$ by the fourth power of the greatest common divisor of x and y leads to $uv(u^2-uv+v^2) = w^2$ for coprime u and v. Because u^2-uv+v^2 is coprime with both u and v, u and v must be square numbers.

5

When x and y are coprime and $x^2+y^2=z^2$ then x and y cannot be both odd. Let y be even. Then for some coprime u and v, $\frac{u}{v} = \frac{z}{y} + \frac{x}{y}$ and $\frac{v}{u} = \frac{z}{y} - \frac{x}{y}$ from $y^2 = (z+x)(z-x)$. Then $\frac{u^2-v^2}{2uv} = \frac{x}{y}$. If u and v are both odd then y cannot be even, so one is odd, the other is even, $x = u^2-v^2$ and $y = 2uv$.

6

When x and y are distinct coprime that satisfies $x^4 - x^2y^2 + y^4 = (x^2 - y^2)^2 + (xy)^2 = z^2$ with minimal xy, let x be odd and y be even. Then $x^2 - y^2 = u^2 - v^2$ and $xy = 2uv$ for some coprime u and v. From the prior equation, u is odd and v is even. Then from the latter equation, $x = ab$, $y = 2cd$, $u = ac$ and $v = bd$ for some coprime a, b, c and d with d being even. Then $(ab)^2 - (2cd)^2 = (ac)^2 - (bd)^2$ leads to $b^2(a^2 + d^2) = c^2(a^2 + 4d^2)$. Because $a^2 + d^2$ is not divided by 3, $a^2 + d^2$ and $a^2 + 4d^2 = (a^2 + d^2) + 3d^2$ are coprime. So $b^2 = a^2 + 4d^2 = a^2 + (2d)^2$ and $a^2 + d^2 = c^2$. Then from the prior equation, $a = m^2 - n^2$ and $d = mn$ for coprime m and n with one being even and the other being odd. Then $(m^2 - n^2)^2 + (mn)^2 = c^2$, but $mn = d < 2cd = y \le xy$ violates the first minimal condition. Let x and y are both odd, then $x^2 - y^2 = 2uv$ and $xy = u^2 - v^2$ for some coprime u and v with one being even and the other being odd. Then $(u^2 - v^2)^2 + (uv)^2 = (xy)^2 + (\frac{x^2-y^2}{2})^2 = (\frac{x^2+y^2}{2})^2$ but there are no such u and v as seen above.

Reference

Pocklington, Some Diophantine Impossibilities, page 111. Retrieved from Oliver Knill's homepage.

Proof of Numerosity

There are numerous natural numbers x and y that make $1 + \frac{x^3+y^3}{1/x+1/y}$ a square number.

$$\frac{2uv}{1 - u + u^2 - 4uv^4} = \sum_{i=1}^{\infty} a_i(v)u^i \tag{1}$$

$$a_0(v) = 0 \tag{2}$$

$$a_1(v) = 2v \tag{3}$$

$$a_{i+1}(v) + a_{i-1}(v) = (4v^4 + 1)a_i(v) \tag{4}$$

$$b_i(v) = a_i^2(v) + a_{i+1}^2(v) - (4v^4 + 1)a_i(v)a_{i+1}(v) \tag{5}$$

$$b_i(v) - b_{i-1}(v) = 0 \tag{6}$$

$$b_0(v) = 4v^2 \tag{7}$$

$$1 + \frac{a_i^3 + a_{i+1}^3}{1/a_i + 1/a_{i+1}} = 1 + a_i a_{i+1}(a_i^2 - a_i a_{i+1} + a_{i+1}^2) \tag{8}$$

$$= 1 + a_i a_{i+1}(b_i + 4v^4 a_i a_{i+1}) = (2v^2 a_i a_{i+1} + 1)^2 \tag{9}$$

Proof of Existence

There are natural numbers x and y that make $2 + \frac{x^3+y^3}{1/x+1/y}$ a square number.

$$\{x, y\} = \{2 \times 47 \times 79 \times 13799, 11801 \times 24121\} \tag{10}$$

$$2 + \frac{x^3 + y^3}{1/x + 1/y} = 42648710892754998^2 \tag{11}$$