Guide to Royden and Fitzpatrick's Real Analysis: The Radon-Nikodym Theorem (Part 1)

written by Peter Burgos on Functor Network
original link: https://functor.network/user/897/entry/385

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space; that is, X may be written as a countable union of measurable sets of finite μ-measure. Let ν be another σ-finite measure on $(X, \mathcal{M})$. Further suppose that μ and ν are related to each other by **absolutely continuity**; in particular, suppose μ and ν have the property that

$$\nu(E) = 0 \text{ whenever } \mu(E) = 0 \text{ for all } E \in \mathcal{M}.$$

Should this relationship be true, we say that ν is **absolutely continuous** with respect to μ, and we denote this by $\nu \ll \mu$. The **Radon-Nikodym Theorem** tells us that there exists a nonnegative measurable function f on X such that

$$\nu(E) = \int_E f \, d\mu \text{ for all } E \in \mathcal{M}., \tag{1}$$

and that this function f is unique up to pointwise almost everywhere equivalence.

The Radon-Nikodym theorem is in the same "flavor" as the Riesz Representation Theorem, in the sense that it suggests that certain abstract functions of interest (in this case, measures) may possess a simpler concrete form (in this case, integration of a fixed function over a measurable set).

My aim in this article is to prove this theorem through a deductive line of reasoning which still makes use of the arguments found in Royden and Fitzpatrick's Real Analysis textbook. I find that by rearranging the arguments in this way, one may find a natural albeit indirect and long path towards our main result which is usually obscured or discarded in favor of the compression and formalization of a proof.

Insights on the recovery of f

Our first goal is to construct a candidate function $\hat{f}$ which we could use as our choice of f in (1). This may be a difficult task using only the abstract properties of measure, so instead, let us try to get insights by tackling a modified form of the problem at hand.

Let us assume that ν may indeed be written in the form in (1) for some nonnegative measurable function f, but no other information concerning f is known. *Using only the values of ν, is there a way to reliably recover f?* To reduce the complexity of this problem, let us assume that $\nu(X) = \int_X f \, d\mu < \infty$.

In essence, through ν, we only know what the integral of f is over arbitrary measurable sets. With this in mind, let us forget ν for now and focus on f. Information on the integrals of f is extremely useful, considering that there are many results that allow us to conclude something about a function knowing only its integrals. To wit, here is a simple example which is a consequence of Chebychev's inequality:

- If g is a measurable function on X, g is nonnegative on $E \in \mathcal{M}$, and $\int_E g = 0$, then $g = 0$ a.e. on E.

This rather simple and intuitive result leads to a very important corollary in the context of our problem.

Corollary Let g_1 be nonnegative and integrable on X. Let g_2 be a nonnegative measurable function on X. If $\int_E g_1\, d\mu \geq \int_E g_2\, d\mu$ for any $E \in \mathcal{M}$, then $g_1 \geq g_2$ a.e. on X.

Proof. Since $\int_X g_2\, d\mu \leq \int_X g_1\, d\mu < \infty$, we conclude g_1 and g_2 are finite a.e. on X hence $g_1 - g_2$ is well-defined and $\int_E g_2\, d\mu < \infty$ for any $E \in \mathcal{M}$.

By way of contradiction, assume $g_1 < g_2$ on some measurable set E' satisfying $\mu(E') > 0$. As a consequence, $\int_{E'} g_1\, d\mu \leq \int_{E'} g_2\, d\mu$. From the conditions of the proposition, we conclude $\int_{E'} g_1\, d\mu = \int_{E'} g_2\, d\mu$, or $\int_{E'} g_2 - g_1\, d\mu = 0$. Hence, $g_2 - g_1 = 0$ a.e. on E'; a contradiction.

By the monotonicity of integration, the converse of the above corollary holds. Thus, we have found a characterization for when a nonnegative integrable function on X dominates another nonnegative measurable function a.e. on X using integrals. With this in mind, let us consider the set

$$\begin{aligned} \mathcal{F} &= \{g : X \to [0, \infty] \mid g \text{ measurable }, g \leq f \text{ a.e. on } X\} \\ &= \left\{ g : X \to [0, \infty] \,\middle|\, g \text{ measurable }, \int_E g \leq \nu(E) = \int_E f \text{ for } E \in \mathcal{M} \right\} \end{aligned} \tag{2}$$

We recall that $\int_X f$ is the supremum of all nonnegative simple functions which f dominates (possibly a.e.) on X. It is not too difficult to show that $\int_X f$ also coincides with the supremum of the integrals of nonnegative *measurable* functions over X which f dominates (possibly a.e) on X. Therefore,

$$\sup_{g \in \mathcal{F}} \int_X g = \int_X f.$$

As a consequence, we can find a sequence $\{g_n\}_{n=1}^{\infty}$ of functions in $\mathcal{F}$ such that $\{\int_X g_n\, d\mu\} \to \int_X f\, d\mu$. If, perchance, the sequence converges pointwise to some function g on X, then we might be getting closer to our goal.

Indeed, since f dominates each function in $\{g_n\}$ a.e. on X, we expect $g \leq f$ or $f - g \geq 0$ a.e. on X. Further, if by some coincidence we find that $\int_X g\, d\mu = \lim_{n\to\infty} \int_X g_n\, d\mu = \int_X f\, d\mu$, then $\int_X f - g\, d\mu = 0$. By applying the "simple example" earlier to $f - g$, we conclude $f - g = 0$ a.e. on X! Thus f is recovered through g. QED.

...But of course, we cannot guarantee such a string of coincidences. So instead, let us construct a sequence that would satisfy our desired properties using $\{g_n\}_{n=1}^{\infty}$ as our building blocks. For each $n \in \mathbb{N}$, let us define $\hat{f}_n = \max\{g_1, g_2, \ldots, g_n\}$. By construction, $\{\hat{f}_n\}$ is increasing, so by the Monotone Convergence Theorem it converges pointwise to some nonnegative measurable function $\hat{f}$ on X. In addition,

$$\int_X \hat{f} = \lim_{n\to\infty} \int_X \hat{f}_n.$$

Now, for $n \in \mathbb{N}$, we know that the functions $g_1, \ldots, g_n$ are each dominated a.e. on X by f. As a consequence, we expect $\hat{f}_n = \max\{g_1, \ldots, g_n\}$ to be dominated a.e. on X by f as well, i.e., $\hat{f}_n \leq f$ a.e. on X. Therefore, the pointwise limit $\hat{f}$ of $\{f_n\}$ must be dominated a.e. on X by f, i.e., $\hat{f}_n \leq f$ a.e. on X. By the monotonocity of integration, $\int_X \hat{f}\, d\mu \leq \int_X f\, d\mu$.

In addition, since $\hat{f}_n \geq g_n$ by our construction of f_n, we find that

$$\int_X \hat{f}\, d\mu = \lim_{n\to\infty} \hat{f}_n\, d\mu \geq \lim_{n\to\infty} g_n = \int_X f\, d\mu.$$

We could thus conclude that $\int_X \hat{f}\, d\mu = \int_X f\, d\mu$. Now, recall that $\hat{f} \leq f$ a.e. on X. By applying our earlier mentioned "simple result" to $f - \hat{f}$ on X, we conclude that $f - \hat{f} = 0$ a.e. on X. That is, $\hat{f} = f$ a.e. on X.

We have now recovered f! To summarize, here are the main steps which we employed:

1. Consider the set

   $$\mathcal{F} = \left\{ g : X \to [0, \infty] \mid g \text{ measurable}, \int_E g\, d\mu \leq \int_E f\, d\mu \text{ for } E \in \mathcal{M} \right\}$$

   and get a sequence $\{g_n\}$ in $\mathcal{F}$ such that

   $$\int_X g_n\, d\mu \to \sup_{g\in\mathcal{F}} \int_X g\, d\mu = \int_X f\, d\mu.$$

2. From $\{g_n\}$, construct a new sequence $\{\hat{f}_n\}$ where $\hat{f}_n = \max\{g_1, \ldots, g_n\}$. Let $\hat{f}$ be the pointwise limit function of this sequence which is guaranteed to exist by the Monotone Convergence Theorem.

3. Show that $\hat{f} \leq f$ a.e. on X and that $\int_X \hat{f} = \int_X f$, thereby conclude that $\hat{f} = f$ a.e. on X.

We have thus found a procedure to get f using only its integrals. Using this as our guide, in the next article, we shall replace the integration of f with the evaluation of an arbitrary measure, and see which among the steps above could be retained or modified in order to prove the Radon-Nikodym theorem.