

Guide to Royden and Fitzpatrick's Real Analysis: The Radon-Nikodym Theorem (Part 1)

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Let (X, \mathcal{M}, μ) be a σ -finite measure space; that is, X may be written as a countable union of measurable sets of finite μ -measure. Let ν be another σ -finite measure on (X, \mathcal{M}) . Further suppose that μ and ν are related to each other by **absolutely continuity**; in particular, suppose μ and ν have the property that

$$\nu(E) = 0 \text{ whenever } \mu(E) = 0 \text{ for all } E \in \mathcal{M}.$$

Should this relationship be true, we say that ν is **absolutely continuous** with respect to μ , and we denote this by $\nu \ll \mu$. The **Radon-Nikodym Theorem** tells us that there exists a nonnegative measurable function f on X such that

$$\nu(E) = \int_E f d\mu \text{ for all } E \in \mathcal{M}, \tag{1}$$

and that this function f is unique up to pointwise almost everywhere equivalence.

The Radon-Nikodym theorem is in the same “flavor” as the Riesz Representation Theorem, in the sense that it suggests that certain abstract functions of interest (in this case, measures) may possess a simpler concrete form (in this case, integration of a fixed function over a measurable set).

My aim in this article is to prove this theorem through a deductive line of reasoning which still makes use of the arguments found in Royden and Fitzpatrick's Real Analysis textbook. I find that by rearranging the arguments in this way, one may find a natural albeit indirect and long path towards our main result which is usually obscured or discarded in favor of the compression and formalization of a proof.

Insights on the recovery of f

Our first goal is to construct a candidate function \hat{f} which we could use as our choice of f in (1). This may be a difficult task using only the abstract properties of measure, so instead, let us try to get insights by tackling a modified form of the problem at hand.

Let us assume that ν may indeed be written in the form in (1) for some nonnegative measurable function f , but no other information concerning f is known. *Using only the values of ν , is there a way to reliably recover f ?* To reduce the complexity of this problem, let us assume that $\nu(X) = \int_X f d\mu < \infty$.

In essence, through ν , we only know what the integral of f is over arbitrary measurable sets. With this in mind, let us forget ν for now and focus on f . Information on the integrals of f is extremely useful, considering that there are many results that allow us to conclude something about a function knowing only its integrals. To wit, here is a simple example which is a consequence of Chebychev's inequality:

- If g is a measurable function on X , g is nonnegative on $E \in \mathcal{M}$, and $\int_E g = 0$, then $g = 0$ a.e. on E .

This rather simple and intuitive result leads to a very important corollary in the context of our problem.

Corollary Let g_1 be nonnegative and integrable on X . Let g_2 be a nonnegative measurable function on X . If $\int_E g_1 d\mu \geq \int_E g_2 d\mu$ for any $E \in \mathcal{M}$, then $g_1 \geq g_2$ a.e. on X .

Proof. Since $\int_X g_2 d\mu \leq \int_X g_1 d\mu < \infty$, we conclude g_1 and g_2 are finite a.e. on X hence $g_1 - g_2$ is well-defined and $\int_E g_2 d\mu < \infty$ for any $E \in \mathcal{M}$.

By way of contradiction, assume $g_1 < g_2$ on some measurable set E' satisfying $\mu(E') > 0$. As a consequence, $\int_{E'} g_1 d\mu < \int_{E'} g_2 d\mu$. From the conditions of the proposition, we conclude $\int_{E'} g_1 d\mu = \int_{E'} g_2 d\mu$, or $\int_{E'} g_2 - g_1 d\mu = 0$. Hence, $g_2 - g_1 = 0$ a.e. on E' ; a contradiction. \square

By the monotonicity of integration, the converse of the above corollary holds. Thus, we have found a characterization for when a nonnegative integrable function on X dominates another nonnegative measurable function a.e. on X using integrals. With this in mind, let us consider the set

$$\begin{aligned} \mathcal{F} &= \{g : X \rightarrow [0, \infty] \mid g \text{ measurable, } g \leq f \text{ a.e. on } X\} \\ &= \left\{ g : X \rightarrow [0, \infty] \mid g \text{ measurable, } \int_E g \leq \nu(E) = \int_E f \text{ for } E \in \mathcal{M} \right\} \end{aligned} \quad (2)$$

We recall that $\int_X f$ is the supremum of all nonnegative simple functions which f dominates (possibly a.e.) on X . It is not too difficult to show that $\int_X f$ also coincides with the supremum of the integrals of nonnegative *measurable* functions over X which f dominates (possibly a.e.) on X . Therefore,

$$\sup_{g \in \mathcal{F}} \int_X g = \int_X f.$$

As a consequence, we can find a sequence $\{g_n\}_{n=1}^{\infty}$ of functions in \mathcal{F} such that $\{\int_X g_n d\mu\} \rightarrow \int_X f d\mu$. If, perchance, the sequence converges pointwise to some function g on X , then we might be getting closer to our goal.

Indeed, since f dominates each function in $\{g_n\}$ a.e. on X , we expect $g \leq f$ or $f - g \geq 0$ a.e. on X . Further, if by some coincidence we find that $\int_X g d\mu = \lim_{n \rightarrow \infty} \int_X g_n d\mu = \int_X f d\mu$, then $\int_X f - g d\mu = 0$. By applying the “simple example” earlier to $f - g$, we conclude $f - g = 0$ a.e. on X ! Thus f is recovered through g . QED.

...But of course, we cannot guarantee such a string of coincidences. So instead, let us construct a sequence that would satisfy our desired properties using $\{g_n\}_{n=1}^{\infty}$ as our building blocks. For each $n \in \mathbb{N}$, let us define $\hat{f}_n = \max\{g_1, g_2, \dots, g_n\}$. By construction, $\{\hat{f}_n\}$ is increasing, so by the Monotone Convergence Theorem it converges pointwise to some nonnegative measurable function \hat{f} on X . In addition,

$$\int_X \hat{f} = \lim_{n \rightarrow \infty} \int_X \hat{f}_n.$$

Now, for $n \in \mathbb{N}$, we know that the functions g_1, \dots, g_n are each dominated a.e. on X by f . As a consequence, we expect $\hat{f}_n = \max\{g_1, \dots, g_n\}$ to be dominated a.e. on X by f as well, i.e., $\hat{f}_n \leq f$ a.e. on X . Therefore, the pointwise limit \hat{f} of $\{\hat{f}_n\}$ must be dominated a.e. on X by f , i.e., $\hat{f} \leq f$ a.e. on X . By the monotonicity of integration, $\int_X \hat{f} d\mu \leq \int_X f d\mu$.

In addition, since $\hat{f}_n \geq g_n$ by our construction of \hat{f}_n , we find that

$$\int_X \hat{f} d\mu = \lim_{n \rightarrow \infty} \int_X \hat{f}_n d\mu \geq \lim_{n \rightarrow \infty} \int_X g_n d\mu = \int_X f d\mu.$$

We could thus conclude that $\int_X \hat{f} d\mu = \int_X f d\mu$. Now, recall that $\hat{f} \leq f$ a.e. on X . By applying our earlier mentioned “simple result” to $f - \hat{f}$ on X , we conclude that $f - \hat{f} = 0$ a.e. on X . That is, $\hat{f} = f$ a.e. on X .

We have now recovered f ! To summarize, here are the main steps which we employed:

1. Consider the set

$$\mathcal{F} = \left\{ g : X \rightarrow [0, \infty] \mid g \text{ measurable, } \int_E g d\mu \leq \int_E f d\mu \text{ for } E \in \mathcal{M} \right\}$$

and get a sequence $\{g_n\}$ in \mathcal{F} such that

$$2. \quad \int_X g_n d\mu \rightarrow \sup_{g \in \mathcal{F}} \int_X g d\mu = \int_X f d\mu.$$

From $\{g_n\}$, construct a new sequence $\{\hat{f}_n\}$ where $\hat{f}_n = \max\{g_1, \dots, g_n\}$. Let \hat{f} be the pointwise limit function of this sequence which is guaranteed to exist by the Monotone Convergence Theorem.

3. Show that $\hat{f} \leq f$ a.e. on X and that $\int_X \hat{f} = \int_X f$, thereby conclude that $\hat{f} = f$ a.e. on X .

We have thus found a procedure to get f using only its integrals. Using this as our guide, in the next article, we shall replace the integration of f with the evaluation of an arbitrary measure, and see which among the steps above could be retained or modified in order to prove the Radon-Nikodym theorem.