PE 890 (Spoilers)

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An ongoing exploration into the binary function (and possibly some generalisations)

The goal post

The binary partition function is denoted as $p(n)$ and is the number of ways to split n into a sum of powers of two where order of summands does not matter. We want to find $p(a^b)$ modulo some prime where $b \approx 1000$.

Aimless Wandering

Observations:

1. If n is odd, then $p(n) = p(n-1)$.
2. $p(n)$ is the $\lfloor \frac{n}{2} \rfloor$-th prefix sum of the sequence $p(0), p(1), p(2), \ldots$
3. From (2) and (1), if we take the prefix sum of the sequence, we now get every even (or odd) term.
4. $p(n+2) = p(n) + p(\lfloor \frac{n}{2} \rfloor)$ or something like that.

At this point, I'm highly suspicious about (4) because if we take the prefix sum of the prefix sum, we get a sequence which behaves almost like (4).

$p_0 : 1\ 1\ 2\ 2\ 4\ 4\ 6\ 6\ 10\ 10\ 14\ 14\ 20\ 20\ 26\ 26\ 36\ 36\ 46\ 46\ 60$

$p_1 : 1\ 2\ 4\ 6\ 10\ 14\ 20\ 26\ 36\ 46\ 60\ 74\ 94\ 114\ 140\ 166\ 202\ 238\ 284\ 330\ 390$

$p_2 : 1\ 3\ 7\ 13\ 23\ 37\ 57\ 83\ 119\ 165\ 225\ 299\ 393\ 507\ 647\ 813\ 1015\ 1253\ 1537\ 1867\ 2257$

These are the terms and their respective prefix sums. We denote then $p_0, p_1, p_2, \ldots$ In general, we refer to the sequence obtained by taking prefix sums k times as p_k. What I mean above, is notice that in p_2, we have $7 = 3 + (3+1), 13 = 7 + (3+3), 23 = 13 + (7+3), 37 = 23 + (7+7)$. Which is not exactly (4), but it has the same flavor as (4).

So maybe, this pattern continues?

1 1 2 2 4 4 6 6 10 10 14
1 2 4 6 10 14 20 26 36 46 60
1 3 7 13 23 37 57 83 119 165 225
1 4 11 24 47 84 141 224 343 508 733
1 5 16 40 87 171 312 536 879 1387 2120
1 6 22 62 149 320 632 1168 2047 3434 5554
1 7 29 91 240 560 1192 2360 4407 7841 13395
1 8 37 128 368 928 2120 4480 8887 16728 30123
1 9 46 174 542 1470 3590 8070 16957 33685 63808
1 10 56 230 772 2242 5832 13902 30859 64544 128352

Indeed it does! $11 = 4 + (3 + 4), 24 = 11 + (3 + 1), 47 = 24 + (3 + 11), 84 = 47 + (3 + 4), … $ Okay, but this seems completely random and out of this world and also, it doesn't give us any information on our original sequence.

If we study, the relation between sequence, we can see that between p_0 and p_1 we have that there is a one to one correspondence between even terms of the former and terms of the latter. The relation holds for p_1 and p_2 except all the terms are doubled. This seems to fail for p2 and p_3, but if we wish hard enough, we find $p_2(2n) = 3p_3(n) + p_3(n-1)$.

Writing a program to figure out this linear relation…

1
2
3 1
4 4
5 10 1
6 20 6
7 35 21 1
8 56 56 8
9 84 126 36 1

These are the coefficients of the linear relations. Damn, they are the binomial coefficients. We're still missing half the terms if we want to work backwards. But miracuously they also follow some linear relation with binomial coefficients.

Maybe it isn't that miraculous? There is definitely a way to proceed via generating functions. I retire to sleep for now. Proof coming soon TM