Abelianization Volley #2 - Abelianization

Tucker Q • 18 May 2024

Welcome back to this series on abelianization and category theory! In case you're new here, the content of this post depends on my last post and some basic group theory. This time we'll be discussing the titular concept, abelianization, and formulating it in terms of universal arrows.

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First, a quick note on notation for groups. I'll be writing the operation in an arbitrary group as juxtaposition, using e to denote the identity element, and a^{-1} to denote the inverse of a. When discussing quotient groups, I'll be using right cosets. With this clarified, we can introduce the definition which will be central to this discussion.

Definition 1. Let G be a group, and let a and b be elements of G. The commutator of a and b is the element $aba^{-1}b^{-1}$, also denoted $[a, b]$.

[1]

The commutator of two elements can "detect" whether those elements commute, as $[a, b] = e$ if and only if $ab = ba$. Therefore, an abelian group can also be defined as a group in which all commutators are the identity. In a non-abelian group G, commutators can be used to study how "far" G is from being abelian. Informally, the more nonidentity commutators a group has, the "further" it is from being abelian. One of the goals of this post is to make this intuitive idea more rigorous. To begin with this, we'll need one more definition.

Definition 2. Let G be a group. The commutator subgroup, or derived subgroup, of G is the subgroup generated by all the commutators of pairs of elements in G. It is denoted $[G, G]$.

It should be clear that for any abelian group G, $[G, G] = \{e\}$. For a more interesting example, consider the smallest nonabelian group, S_3. We could compute $[S_3, S_3]$ by checking the commutator for every single pair of elements, but this would be a bit too much effort. Instead, there is a more direct approach due to Gary Myerson on MSE. For permutations a and b, the commutator $[a, b] = aba^{-1}b^{-1}$ must be an even permutation, as the parity of a permutation and its inverse match. In S_3, the even permutations are the identity and the 3-

cycles, which together comprise the alternating group A_3. Because all commutators are even, $[S_3, S_3]$ must be a subgroup of A_3. But $[S_3, S_3]$ cannot be trivial, and so $[S_3, S_3] = A_3$.

Some groups are their own commutator subgroups; these groups are called "perfect groups". The smallest nontrivial perfect group is A_5. I've heard rumors that this is why arbitrary quintic equations cannot be solved by radicals, but I don't know enough Galois theory to corroborate those tales; perhaps I'll post about it here later if I have time to explore this further. At any rate, examples are useful for cutting our teeth, but in true categorical fashion, we're more interested in chasing after general results. Here is one that will be very useful.

Lemma 1. *For any group G, the commutator subgroup $[G, G]$ is normal.*

Proof. Let $g \in G$ and $n \in [G, G]$; we must demonstrate that $gng^{-1} \in [G, G]$. Since $[G, G]$ contains all commutators, it contains $[g, n]$. And since it is closed under the group operation, $[G, G]$ contains $[g, n]n = gng^{-1}n^{-1}n = gng^{-1}$. □

This can be helpful for ruling out candidates for the commutator subgroup, as non-normal subgroups can immediately be eliminated. It also shows us that every nonabelian simple group must be a perfect group, although there are also nonsimple perfect groups. However, the really useful part of this theorem is that it means we can consider the quotient group $G/[G, G]$. But before we begin considering this group, we will prove the following theorem which explains why this group is of particular interest.

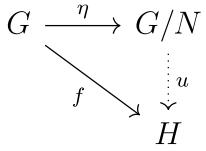
Theorem 2. *Let G be a group and N a normal subgroup of G. The quotient group G/N is abelian if and only if N contains $[G, G]$.*

Proof. First, suppose the quotient group G/N is abelian, and let a, b be elements of G. Since G/N is abelian, $(Na)(Nb) = (Nb)(Na) \Rightarrow N(ab) = N(ba)$. Since ab and ba are in the same coset, there exists some $n \in N$ such that $ab = nba$. We can right-multiply by $a^{-1}b^{-1}$ to find $n = aba^{-1}b^{-1} = [a, b]$. Since a and b are arbitrary, this means N contains all of the commutators in G. But since N is a subgroup, it is closed under the group operation, which means it must also contain the entire subgroup $[G, G]$.

Conversely, suppose $[G, G] \subseteq N$, and let a, b be elements of G. Since N contains all commutators, there exists some $n \in N$ such that $n = [a, b] = aba^{-1}b^{-1}$. This time, left-multiplying by ba gives $nba = ab$, meaning ab and ba are in the same coset; $N(ab) = N(ba)$. This means that for arbitrary cosets Na and Nb in G/N, we have $(Na)(Nb) = N(ab) = N(ba) = (Nb)(Na)$, so G/N is abelian. □

This theorem justifies the claim that $[G, G]$ is the smallest normal subgroup of G which gives rise to an abelian quotient group. Conversely, $G/[G, G]$ is the largest abelian quotient group of G. As you may have guessed, we have a special name for this group: the *abelianization* of G, also denoted G^{ab}. Once again, we can consider some examples. Since for any group G, $G \cong G/\{e\}$, every abelian group is isomorphic to its abelianization, which certainly makes sense. For S_3, since the normal subgroup A_3 has an index of 2, $S_3^{\text{ab}} \cong \mathbb{Z}_2$, the cyclic group with two elements. Finally, as G/G is always the trivial group, every perfect group has a trivial abelianization. Now, as you have probably guessed, we will be interested in proving that abelianization satisfies a certain universal property. Before we get there, we will need the following result, which extends the first isomorphism theorem for groups.

Theorem 3. *Let G be a group, N be a normal subgroup of G, $f : G \to H$ be a group homomorphism with $N \subseteq \ker(f)$, and $\eta : G \to G/N$ the canonical homomorphism associated with the quotient. Then there is a unique group homomorphism $u : G/N \to H$ such that the following diagram commutes.*



(1)

That is, such that $u \circ \eta = f$.

Proof. Define a mapping $u : G/N \to H$ by $u(Na) = f(a)$. Since this function is defined in terms of cosets, we must first show that it is indepenent of the choice of equivalence class representative. Let g_1 and g_2 be elements of G which belong to the same coset. Then there is an element $n \in N$ such that $g_1 = ng_2$. We need to show $u(Ng_1) = u(Ng_2)$. Since $n \in \ker(f)$, this is rather simple:

$$u(Ng_1) = f(g_1) = f(ng_2) = f(g_2) = u(Ng_2) \tag{2}$$

Next, we must show that u is a group homomorphism. For this, let Ng and Nh be two not necessarily equal cosets. Then we have:

$$u(NgNh) = u(Ngh) = f(gh) = f(g)f(h) = u(Ng)u(Nh) \tag{3}$$

Next, it is simple to see that u makes the diagram commute, as $(u \circ \eta)(g) = u(\eta(g)) = u(Ng) = f(g)$. We must finally demonstrate that u is unique relative to this property. To see this, suppose we have another homomorphism $v : G/N \to H$ such that $v \circ \eta = f$. Then for any coset $Ng \in G/N$, $v(Ng) = v(\eta(g)) = (v \circ \eta)(g) = f(g) = u(Ng)$, so $u = v$. $\square$

This theorem, often referred to as the universal property of the quotient group, will be key in proving the result we are really after in this post: the universal property satisfied by a group's abelianization.

Theorem 4. *Let G be a group, $\eta : G \to G^{\text{ab}}$ the canonical homomorphism associated with the quotient $G/[G,G]$, and H be an abelian group. Then for any homomorphism $f : G \to H$, there is a unique homomorphism $u : G^{\text{ab}} \to H$ such that the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\eta} & G^{\text{ab}} \\ & \searrow^{f} & \downarrow{\scriptstyle u} \\ & & H \end{array} \tag{4}$$

Proof. Carrying on from the previous theorem, it suffices to show that $[G,G]$ is contained in $\ker(f)$. Since $[G,G]$ is generated by the commutators of pairs of elements in G, it suffices to show that for any elements $a, b \in G$, $[a,b] \in \ker(f)$. That is, we want to show that $f(aba^{-1}b^{-1}) = e$, or equivalently, that $f(ab) = f(ba)$. Since H is abelian, this is rather simple:

$$f(ab) = f(a)f(b) = f(b)f(a) = f(ba) \tag{5}$$

□

Now, this theorem is fine and all, but it leaves one question begging to be asked. How can we formulate this as a universal arrow? It has the same shape as the universal arrow diagrams we discussed last time, but we need to introduce a functor. For this, we will consider the inclusion functor $I : \mathbf{Ab} \to \mathbf{Grp}$. We claim that for any group G, the pair (G^{ab}, η) described above is a universal arrow from G to I. To show that this is the case, we must demonstrate that for every abelian group H and group homomorphism $f : G \to I(H)$, there is a unique homomorphism of abelian groups $u : G^{\text{ab}} \to H$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\eta} & I(G^{\text{ab}}) \\ & \searrow^{f} & \downarrow{\scriptstyle I(u)} \\ & & I(H) \end{array} \tag{6}$$

Since I is the inclusion functor, it leaves G^{ab}, u, and H unchanged, so this follows immediately from the theorem we have just proven. This means that (up to isomorphism), the abelianization of a group G may be characterized as a universal arrow from G to the inclusion of $\mathbf{Ab}$ in $\mathbf{Grp}$. That's all I wanted to

discuss this time around; next time, I'll be discussing adjoint functors, a very powerful categorical concept that will tie a lot of the work we've done so far together.

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1. Some sources use define the commutator as $[a, b] = a^{-1}b^{-1}ab$ instead. The particular definition used should not impact the results presented here.↩