

Abelianization Volley #1 - Universal Arrows

Tucker Q • 11 May 2024

Welcome to the first series of posts on this blog! First, I want to describe the motivation behind this series of posts and what it will attempt to do. This past semester, I took a specialized course on category theory. It was a small group with a fantastic professor, and it was one of the most enjoyable courses I've taken. Even so, there were some concepts that were difficult to follow, namely universal arrows¹ and adjoint functors. I managed to wrap my head around these topics enough to do well on the final exam, but I still feel my understanding could be improved. So one goal of the present series of posts is to explore those concepts more thoroughly, trying to explain them in my own words and justify them rather than just stating definitions and proving theorems. The other goal of this series is to address the titular term “abelianization”. I've been self-studying some more algebra this summer as my school only offers a one-semester course, and thanks to Wikipedia, I learned that abelianization provides an interesting example of universal arrows and adjoint functors. Thus, the second goal of this “volley” of posts is to describe abelianization in the context of category theory.

Given that these are the goals of this series, I have to make some pretty heavy assumptions of pre-existing understanding in order to write these posts without spending 75% or more of my time on buildup. On the category theory side, I assume familiarity with categories, basic categorical constructions, functors, and natural transformations. On the algebra side, I assume familiarity with basic group theory, including homomorphisms, kernels, normal subgroups, and quotient groups. Now, with the housekeeping out of the way, let's get on to the math.

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In this first post, I'll introduce the concept of universal arrows, give some examples of them, and prove some basic theorems. Many times in category theory, or algebra more broadly, we encounter so-called “universal constructions,” which usually induce certain unique arrows according to certain properties. The principle behind the idea of a “universal arrow” is to provide a general framework for this concept.

Definition 1. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, $F : \mathbf{C} \rightarrow \mathbf{D}$ be a functor, and $A \in \text{ob}(\mathbf{D})$. A universal arrow from A to F is an ordered pair (R, u) , where $R \in \text{ob}(\mathbf{C})$ and $u \in \text{Hom}_{\mathbf{D}}(A, F(R))$. This pair must satisfy the additional condition that for every pair (B, f) , where $B \in \text{ob}(\mathbf{C})$ and $f \in \text{Hom}_{\mathbf{D}}(A, F(B))$, there is a unique arrow $f' : R \rightarrow B$ in $\mathbf{C}$ such that $F(f') \circ u = f$. Put diagrammatically, the arrow f' is the unique one such that the following diagram commutes.

$$\begin{array}{ccc}
 R & & A \xrightarrow{u} F(R) \\
 \vdots f' & \searrow f & \vdots F(f') \\
 B & & F(B)
 \end{array} \tag{1}$$

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To wrap our heads around this incredibly abstract definition, we'll consider a familiar construction, the coproduct, as a universal arrow. In order to do this, we'll need to define the diagonal functor.

Definition 2. Let $\mathbf{C}$ be a category. The diagonal functor is a functor $\Delta : \mathbf{C} \rightarrow \mathbf{C} \times \mathbf{C}$ that “doubles up” objects and arrows. More specifically, for $A \in \text{ob}(\mathbf{C})$, $\Delta(A) = (A, A)$, and for $f : A \rightarrow B$ in $\mathbf{C}$, $\Delta(f) = (f, f)$.

Now, let A and B be objects in $\mathbf{C}$ which have a coproduct $A \amalg B$. Let $\iota_A : A \rightarrow A \amalg B$ and $\iota_B : B \rightarrow A \amalg B$ be the coprojection arrows associated with the coproduct. We will show that the pair $(A \amalg B, (\iota_A, \iota_B))$ is a universal arrow from (A, B) to Δ . To demonstrate this, suppose we have an arrow $(f, g) : (A, B) \rightarrow (X, X)$ in $\mathbf{C} \times \mathbf{C}$, noting that $\Delta(X) = (X, X)$. We need to show that there is a unique arrow $u : A \amalg B \rightarrow X$ such that the following diagram commutes.

$$\begin{array}{ccc}
 A \amalg B & & (A, B) \xrightarrow{(\iota_A, \iota_B)} \Delta(A \amalg B) \\
 \vdots u & \searrow (f, g) & \vdots \Delta(u) \\
 X & & \Delta(X)
 \end{array} \tag{2}$$

The definition of the coproduct already gives us this exact arrow: a unique arrow $u : A \amalg B \rightarrow X$ such that $u \circ \iota_A = f$ and $u \circ \iota_B = g$. This in turn means that $(u, u) \circ (\iota_A, \iota_B) = (f, g)$ as desired. By following a similar line of logic, it should be easy to see how every universal arrow from (A, B) to Δ satisfies the definition of a coproduct for A and B . This sort of correspondence leads us nicely to the following property of universal arrows:

Theorem 1. Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a functor and A an object in $\mathbf{D}$. Then universal arrows from A to F are unique up to a unique isomorphism. In detail, if (R, u) and (R', u') are both universal arrows from A to F , there is a unique isomorphism $\alpha : R \rightarrow R'$ such that $F(\alpha) \circ u = u'$.

Proof. By the definition of a universal arrow, we already have a unique arrow $\alpha : R \rightarrow R'$ such that $F(\alpha) \circ u = u'$ and another unique arrow $\alpha' : R' \rightarrow R$ such that $F(\alpha') \circ u' = u$. These arrows are uniquely specified by the following diagrams.

$$\begin{array}{ccc}
 A & \xrightarrow{u} & F(R) \\
 & \searrow u' & \downarrow F(\alpha) \\
 & & F(R')
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{u'} & F(R') \\
 & \searrow u & \downarrow F(\alpha') \\
 & & F(R)
 \end{array}
 \tag{3}$$

We will show that α and α' are inverses. To do this, note that the universality of u guarantees a unique arrow $x : R \rightarrow R$ such that $F(x) \circ u = u$. Clearly 1_R satisfies this property, since $F(1_R) = 1_{F(R)}$. But $\alpha' \circ \alpha$ satisfies the same property, since

$$F(\alpha' \circ \alpha) \circ u = F(\alpha') \circ F(\alpha) \circ u = F(\alpha') \circ u' = u \tag{4}$$

This means we must have $\alpha' \circ \alpha = 1_R$. Similarly, the universality of u' guarantees a unique arrow $y : R' \rightarrow R'$ such that $u' \circ F(y) = u'$. Again, $1_{R'}$ clearly fills this role. But so does $\alpha \circ \alpha'$, since

$$F(\alpha \circ \alpha') \circ u' = F(\alpha) \circ F(\alpha') \circ u' = F(\alpha) \circ u = u' \tag{5}$$

This means $\alpha \circ \alpha' = 1_{R'}$, showing that α is an isomorphism as desired. $\square$

This theorem should help assure us that universal arrows are in fact a good way of describing the universal constructions we tend to encounter in mathematics, as uniqueness up to isomorphism is a hallmark of these objects. But part of the real magic of universal arrows is seeing how they show up all over the place. We have already seen how the coproduct arises as a universal arrow, but that's a bit of a tame example. This next example will be more interesting, as we have to put in more work to fulfill the definition of a universal arrow.

Before we go on, I should warn that this example was not an original part of the plan for this post, and it might get a bit in the weeds on the linear algebra side of things. However, once I got started, I was too excited not to include it, so I hope you'll indulge me.

Definition 3. Let V and W be vector spaces over a fixed field K . Then their tensor product, if it exists, is a vector space $V \otimes W$ equipped with a bilinear map $\phi : V \times W \rightarrow V \otimes W$ such that for all bilinear maps $\theta : V \times W \rightarrow X$, there is a unique linear map u which makes the following diagram commute.

$$\begin{array}{ccc} V \times W & \xrightarrow{\phi} & V \otimes W \\ & \searrow \theta & \downarrow u \\ & & X \end{array} \quad (6)$$

Looking at this definition, it seems to scream “universal arrow”. It has the exact same shape as the universal arrow diagrams, and most sources describe this definition as the universal property of the tensor product. But at the same time, it kind of looks like a mess; we have linear maps and bilinear maps mixed together in the same commutative diagram, and there are no functors in sight. As it turns out, dear reader, finding how functors fit in and make this diagram into something cleaner will be incredibly beautiful. In this next section, I should acknowledge the help of Wikipedia’s discussion of the [Tensor-Hom Adjunction](#) in guiding my thought process. I know, mentioning adjunctions is spoilers for later posts in this volley, but perhaps this hints at how deep the connection between universal arrows and adjunctions runs.

Our first goal will be to turn the bilinear maps into linear maps. Recall that if A and B are vector spaces, the set $\text{Hom}(A, B)$ of linear maps from A to B also forms a vector space under the elementwise operations, i. e.:

$$(f + g)(x) = f(x) + g(x); (af)(x) = a[f(x)] \quad (7)$$

With this in mind, we will show that bilinear maps from $V \times W$ to an arbitrary vector space X correspond bijectively with linear maps from V to $\text{Hom}(W, X)$. First, suppose $\theta : V \times W \rightarrow X$ is a bilinear map. By the definition of a bilinear map, each $v \in V$ determines a linear map $\theta(v, -) : W \rightarrow X$. Now define a mapping $c : V \rightarrow \text{Hom}(W, X)$ by $c(v) = \theta(v, -)$. To demonstrate that c is a linear map, we will show that $c(av_1 + v_2)(w) = a[c(v_1)(w)] + c(v_2)(w)$ for all $a \in K$, $v_1, v_2 \in V$, and $w \in W$:

$$c(av_1 + v_2)(w) = \theta(av_1 + v_2, w) = a[\theta(v_1, w)] + \theta(v_2, w) = a[c(v_1)(w)] + c(v_2)(w) \quad (8)$$

For the opposite direction, suppose we begin with a linear map $c : V \rightarrow \text{Hom}(W, X)$. Define a mapping $\theta : V \times W \rightarrow X$ by $\theta(v, w) = c(v)(w)$. We will similarly show that θ is linear in each of its arguments:

$$\theta(av_1+v_2, w) = c(av_1+v_2)(w) = a[c(v_1)(w)]+c(v_2)(w) = a[\theta(v_1, w)]+\theta(v_2, w) \quad (9)$$

$$\theta(v, aw_1+w_2) = c(v)(aw_1+w_2) = a[c(v)(w_1)]+c(v)(w_2) = a[\theta(v, w_1)]+\theta(v, w_2) \quad (10)$$

This gives us a unique way of transforming between bilinear maps and linear maps into the hom-space. This process, called *currying*, will also help us to “functorize” our tensor product problem. First, recall how the covariant hom-functor is usually defined:

Definition 4. Let $\mathbf{C}$ be a locally small category and A an object in $\mathbf{C}$. Then there is a covariant functor $\text{Hom}(A, -) : \mathbf{C} \rightarrow \mathbf{Set}$, which sends an object B to the set $\text{Hom}(A, B)$. It sends an arrow $f : B \rightarrow C$ to the arrow $\text{Hom}(A, f) : \text{Hom}(A, B) \rightarrow \text{Hom}(A, C)$ which is defined by $g \mapsto f \circ g$ for any $g \in \text{Hom}(A, B)$.

The category of vector spaces $\mathbf{Vect}_K$ is rather unique, in that (as we have discussed), $\text{Hom}(A, B)$ is also an object in $\mathbf{Vect}_K$. In addition to this, for a linear map $f : B \rightarrow C$, the mapping $\text{Hom}(A, f)$ is also a linear map. To see this, let $g, h \in \text{Hom}(A, B)$ and $a \in K$. Then we have:

$$\begin{aligned} \text{Hom}(A, f)(ag + h) &= f \circ (ag + h) = f \circ ag + f \circ h = a(f \circ g) + f \circ h \\ &= a\text{Hom}(A, f)(g) + \text{Hom}(A, f)(h) \end{aligned} \quad (11)$$

This means that in the case of $\mathbf{Vect}_K$, we may regard the hom-functor as a functor from $\mathbf{Vect}_K$ to itself, rather than a functor to $\mathbf{Set}$. This is called an *internal hom-functor*. With all this laid as the backdrop, we will describe the tensor product $V \otimes W$ as a universal arrow from V to the internal hom-functor $\text{Hom}(W, -)$. Let $(A, f : V \rightarrow \text{Hom}(W, A))$ be such a universal arrow; we will show how A satisfies the universal property of the tensor product. First, define the bilinear map $\phi : V \times W \rightarrow A$ by $\phi(v, w) = f(v)(w)$ as previously discussed. Now let $\theta : V \times W \rightarrow X$ be an arbitrary bilinear map. In order to show that (ϕ, A) is the tensor product of V and W , we want to produce a unique linear map $u : A \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{\phi} & A \\ & \searrow \theta & \downarrow u \\ & & X \end{array} \quad (12)$$

Once again following from the discussion on currying, let $g : V \rightarrow \text{Hom}(W, X)$ be the linear map defined by $g(v) = \theta(v, -)$. From the universal property of f , we know that there is a unique linear map $u : A \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc}
 A & & V \xrightarrow{f} \text{Hom}(W, A) \\
 \downarrow u & & \searrow g \\
 X & & \text{Hom}(W, X)
 \end{array}
 \quad (13)$$

The commutativity of this diagram tells us that for all $v \in V$, $u \circ f(v) = g(v)$. Now let $v \in V$ and $w \in W$. Using this property, we determine

$$\theta(v, w) = g(v)(w) = [u \circ f(v)](w) = u[f(v)(w)] = u(\phi(v, w)) = (u \circ \phi)(v, w) \quad (14)$$

This means that (ϕ, A) is the tensor product of V and W as desired. A similar, but reversed, argument shows that a pair satisfying the universal property for the tensor product induces a universal arrow from V to $\text{Hom}(W, -)$. Then via theorem 1, we know that the tensor product is unique up to isomorphism. However, it is important to note that we do now know the tensor product actually exists, only that if it does, this uniqueness property holds. While universal arrows are a good framework for proving uniqueness and exploring relations to other categorical concepts (like the internal hom-functor in this example), one must usually return to actual concrete constructions to prove that an object exists which satisfies the universal property. With all that said, I hope this protracted example has helped demonstrate that universal arrows are ubiquitous in mathematics, even though they may sometimes require a little coaxing to reveal themselves. To close out this post, I'd like to prove one final theorem which relates universal arrows to hom-functors. The utility of this theorem will become apparent later on.

Theorem 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally small categories, and $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then*

- (i) *A pair $(R, u : A \rightarrow F(R))$ is a universal arrow from A to F if and only if the function which maps an arrow $f' : R \rightarrow B$ to the arrow $F(f') \circ u : A \rightarrow F(B)$ is a bijection.*
- (ii) *Furthermore, this family of bijections $\text{Hom}_{\mathcal{C}}(R, B) \cong \text{Hom}_{\mathcal{D}}(A, F(B))$ constitutes a natural isomorphism $\varphi : \text{Hom}_{\mathcal{C}}(R, -) \rightarrow \text{Hom}_{\mathcal{D}}(A, F(-))$.*
- (iii) *Conversely, any such natural isomorphism $\varphi : \text{Hom}_{\mathcal{C}}(R, -) \rightarrow \text{Hom}_{\mathcal{D}}(A, F(-))$ is determined in this way by a unique arrow $u : A \rightarrow F(R)$ such that (R, u) is a universal arrow from A to F .*

Proof. For part (i), let $(R, u : A \rightarrow F(R))$ be a universal arrow from A to F . For each $B \in \text{ob}(\mathbf{C})$, let φ_B be the function from $\text{Hom}_{\mathbf{C}}(R, B)$ to $\text{Hom}_{\mathbf{D}}(A, F(B))$ defined by $\varphi_B(f') = F(f') \circ u$. The very definition of a universal arrow means that each arrow in $\text{Hom}_{\mathbf{D}}(A, F(B))$ has a unique preimage under this mapping, so φ_B is a bijection. For part (ii), to show naturality, let $g \in \text{Hom}_{\mathbf{C}}(B, C)$. We must demonstrate that this diagram commutes:

$$\begin{array}{ccc} \text{Hom}_{\mathbf{C}}(R, B) & \xrightarrow{\varphi_B} & \text{Hom}_{\mathbf{D}}(A, F(B)) \\ \text{Hom}_{\mathbf{C}}(R, g) \downarrow & & \downarrow \text{Hom}_{\mathbf{D}}(A, F(g)) \\ \text{Hom}_{\mathbf{C}}(R, C) & \xrightarrow{\varphi_C} & \text{Hom}_{\mathbf{D}}(A, F(C)) \end{array} \quad (15)$$

To see that this diagram commutes, let $f' \in \text{Hom}_{\mathbf{C}}(R, B)$ be an arbitrary arrow. In order for the diagram to commute, we must have $F(g) \circ \varphi_B(f') = \varphi_C(g \circ f')$. Applying the definitions of φ_B and φ_C , this is true if and only if $F(g) \circ F(f') \circ u = F(g \circ f') \circ u$. Because F preserves composition of arrows, this is true.

Finally, to handle part (iii), let $\varphi : \text{Hom}_{\mathbf{C}}(R, -) \rightarrow \text{Hom}_{\mathbf{D}}(A, F(-))$ be a natural isomorphism. We claim that the pair $(R, \varphi_R(1_R))$ is a universal arrow from A to F . To prove this, we must demonstrate that every arrow $f : A \rightarrow F(B)$ has the form $f = F(f') \circ \varphi_R(1_R)$ for a unique arrow $f' : R \rightarrow B$. To show this, suppose we begin with an arbitrary arrow $f' : R \rightarrow B$. Then the naturality of φ means the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}_{\mathbf{C}}(R, R) & \xrightarrow{\varphi_R} & \text{Hom}_{\mathbf{D}}(A, F(R)) \\ \text{Hom}_{\mathbf{C}}(R, f') \downarrow & & \downarrow \text{Hom}_{\mathbf{D}}(A, F(f')) \\ \text{Hom}_{\mathbf{C}}(R, B) & \xrightarrow{\varphi_B} & \text{Hom}_{\mathbf{D}}(A, F(B)) \end{array} \quad (16)$$

By tracing the paths of 1_R through this diagram, we find the following equality:

$$F(f') \circ \varphi_R(1_R) = \varphi_B(f') \quad (17)$$

Because φ_B is a bijection, for each $f : A \rightarrow F(B)$, there is a unique $f' : R \rightarrow B$ such that $f = \varphi_B(f') = F(f') \circ \varphi_R(1_R)$, completing the proof. $\square$

This theorem can be useful for describing the properties of a universal construction in a more direct way. We can see this in action with each of the examples of universal arrows we've considered so far. For the coproduct, the theorem gives us $\text{Hom}_{\mathbf{C}}(A \amalg B, -) \cong \text{Hom}_{\mathbf{C} \times \mathbf{C}}((A, B), \Delta(-))$. This tells us that arrows out of $A \amalg B$ to a fixed object correspond exactly to pairs of arrows from A and B into that same object, which is another way of expressing the

defining property of the coproduct. For the tensor product, we find $\text{Hom}(V \otimes W, -) \cong \text{Hom}(V, \text{Hom}(W, -))$. Since linear maps from V to $\text{Hom}(W, X)$ correspond to bilinear maps from $V \times W$ to X , this tells us that linear maps out of the tensor product correspond to bilinear maps from $V \times W$, which is exactly what the tensor product is meant to do. This theorem will also be incredibly useful once we begin to discuss adjunctions properly rather than coyly avoiding them.

That's all I've got this time! I hope you enjoyed the read. Next time in this series, we'll head over to the realm of group theory and discuss abelianization as a universal arrow.

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1. It seems most people refer to the arrows in a category as morphisms. My professor referred to them as arrows; evidently Saunders Mac Lane used the term *morphism* in a more specific context, and my professor wanted to be consistent with Mac Lane. I don't have particularly strong convictions about it, but the word "arrow" is in all of my notes and seared into my brain, so it's the term I plan to use on this blog.↵
 2. Like most categorical definitions, this one can be dualized. By reversing the direction of the arrows, one obtains the definition of a universal arrow from F to A . Many of the results we prove here for universal arrows to a functor can also be stated for universal arrows from a functor, with suitable adjustment. In the interest of focus, this series will only consider universal arrows to functors.↵