Newmark method

Linyu Zhu · 27 Apr 2024

$$^{t+\Delta t}\dot{U} = ^{t}\dot{U} + \left[(1-\gamma)\cdot {}^{t}\ddot{U} + \gamma \cdot {}^{t+\Delta t}\ddot{U}\right]\Delta t \tag{1}$$

$$^{t+\Delta t}U = ^{t}U + ^{t}\dot{U}\Delta t + \left[(\frac{1}{2}-\beta)\cdot {}^{t}\ddot{U} + \beta \cdot {}^{t+\Delta t}\ddot{U}\right]\Delta t^2 \tag{2}$$

rearrange Eq. 2，可以得到

$$^{t+\Delta t}\ddot{U} = \frac{1}{\beta \Delta t^2}\left[^{t+\Delta t}U - ^{t}U - ^{t}\dot{U}\Delta t - ^{t}\ddot{U}\Delta t^2(\frac{1}{2}-\beta)\right] \tag{3}$$

substitute Eq. 3 into Eq.1，可以得到

$$^{t+\Delta t}\dot{U} = ^{t+\Delta t}U\frac{\gamma}{\beta\Delta t} - ^{t}U\frac{\gamma}{\beta\Delta t} + ^{t}\dot{U}(1-\frac{\gamma}{\beta}) + ^{t}\ddot{U}(1-\frac{\gamma}{2\beta}\Delta t) \tag{4}$$

substitute Eq.3 and Eq.4 into the Eq.5，

$$M \cdot {}^{t+\Delta t}\ddot{U} + C \cdot {}^{t+\Delta t}\dot{U} + K \cdot {}^{t+\Delta t}U = ^{t+\Delta t}R \tag{5}$$

可以得到Eq.6或者Eq.7

$$\begin{aligned} ^{t+\Delta t}U\left(\frac{1}{\beta\Delta t^2}M + \frac{\gamma}{\beta\Delta t}C + K\right) = & ^{t+\Delta t}R + \\ & M\left[^{t}U\frac{1}{\beta\Delta t^2} + ^{t}\dot{U}\frac{1}{\beta\Delta t} + ^{t}\ddot{U}(\frac{1}{2\beta}-1)\right] + \\ & C\left[^{t}U\frac{\gamma}{\beta\Delta t} + ^{t}\dot{U}(\frac{\gamma}{\beta}-1) + ^{t}\ddot{U}(\frac{\gamma}{2\beta}-1)\Delta t\right] \end{aligned} \tag{6}$$

$$\begin{aligned} ^{t+\Delta t}U\left(\frac{1}{\beta\Delta t^2}M + \frac{\gamma}{\beta\Delta t}C + K\right) = & ^{t+\Delta t}R + \\ & ^{t}U\left(\frac{M}{\beta\Delta t^2} + \frac{C\gamma}{\beta\Delta t}\right) + \\ & ^{t}\dot{U}\left(\frac{M}{\beta\Delta t} + C(\frac{\gamma}{\beta}-1)\right) + \\ & ^{t}\ddot{U}\left(M(\frac{1}{2\beta}-1) + C\Delta t(\frac{\gamma}{2\beta}-1)\right) \end{aligned} \tag{7}$$

Exlicit method

初值计算：已知初始位移${}^0U$、初始速度${}^0\dot{U}$和初始载荷${}^0R$，

- 通过Eq.5可以计算${}^0\ddot{U}$ ：

$$
{}^0\ddot{U} = \left({}^0F - {}^0\dot{U}C - {}^0UK\right)/M
$$

由t时刻推$t+\Delta t$时刻：已知由t时刻的位移、速度、加速度和$t+\Delta t$时刻的载荷，

- 通过Eq.6计算$t+\Delta t$时刻的位移；
- 通过Eq.4计算$t+\Delta t$时刻的速度；
- 通过Eq.3计算$t+\Delta t$时刻的加速度；

Implicit method

该方法基于物体在外力作用下的平衡方程，其中R为externally applied nodal point forces, F 为nodal point forces that correspond to the element stresses. F可以通过${}^tU$进行计算，在<finite element procedures in engineering analysis>section 6.3中有讨论。

$$
M{}^t\ddot{U} = {}^tR - {}^tF \tag{8}
$$

时间步进方法的核心是：已知t时刻的解，求解$t+\Delta$时刻满足Eq.9的解。

$$
M{}^{t+\Delta t}\ddot{U} = {}^{t+\Delta t}R - {}^{t+\Delta t}F \tag{9}
$$

因为t时刻的解已知，所以我们将${}^{t+\Delta t}F$表示为Eq.10，其中ΔF为由于element displacement and stress 增量引起的nodal point forces的增量。

$$
{}^{t+\Delta t}F = {}^tF + \Delta F \tag{10}
$$

ΔF的一阶近似为Eq.11，其中${}^tK$ is a tangent stiffness matrix corresponds to the geometric and material conditions at t，见Eq. 12. ΔU是nodal point displacement 的增量。

$$
\Delta F \doteq {}^tK\Delta U \tag{11}
$$

$$
{}^tK = \frac{\partial {}^tF}{\partial {}^tU} \tag{12}
$$

将Eq.11和Eq.10带入到Eq.9中，得到

$$
M{}^{t+\Delta t}\ddot{U} + {}^tK\Delta U = {}^{t+\Delta t}R - {}^tF \tag{13}
$$

对于静态系统，Eq.13退化为Eq.14。如果系统是线性的，即Eq.12中K独立于位移U，那么由Eq.14可以直接结算ΔU，从而由${}^{t+\Delta t}U = {}^{t}U + \Delta U$得到$t+\Delta t$时刻的解${}^{t+\Delta t}U$。

$$ {}^{t}K\Delta U = {}^{t+\Delta t}R - {}^{t}F \tag{14} $$

实际问题中K与U是相关的，多余Eq.11中的近似会造成比较大的误差，因此需要采用Newton-Raphson迭代方法获得较为准确的解。 为了获得更具有普适性的模型，这里直接对动态系统进行分析（省略了damping forces）。牛顿法的处理方式为

$$ M^{t+\Delta t}\ddot{U}^{(k)} + {}^{t+\Delta}K^{(k-1)}\Delta U^{(k)} = {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(k-1)} \tag{15} $$

注意，对比Eq.13和Eq.15，会发现K和F左上角的t变成了$t+\Delta t$，这是因为Eq.13适用的是线性系统，在计算$t+\Delta$时刻的U时，用的是t时刻的矩阵K，通过Eq.12获得。而Eq.15针对的是非线性系统，在计算${}^{t+\Delta}U$的过程中，K不是直接由t时刻的Eq.12获得的，因此，更适合使用上标${}^{t+\Delta}K$表示在计算${}^{t+\Delta}U$的过程中使用的K。F同理。

从第k-1次迭代到k次迭代，有Eq.16.

$$ {}^{t+\Delta t}U^{(k)} = {}^{t+\Delta t}U^{(k-1)} + \Delta U^{(k)} \tag{16} $$

另外，根据trapezoidal time integration, 有关系式Eq.17和Eq.18

$$ {}^{t+\Delta t}U = {}^{t}U + \frac{\Delta t}{2}\left({}^{t}\dot{U} + {}^{t+\Delta t}\dot{U}\right) \tag{17} $$

$$ {}^{t+\Delta t}\dot{U} = {}^{t}\dot{U} + \frac{\Delta t}{2}\left({}^{t}\ddot{U} + {}^{t+\Delta t}\ddot{U}\right) \tag{18} $$

结合Eq.16, Eq.17和 Eq.18可以得到Eq.19.

$$ {}^{t+\Delta t}\ddot{U}^{(k)} = \frac{4}{\Delta t^2}\left({}^{t+\Delta t}U^{(k-1)} - {}^{t}U + \Delta U^{(k)}\right) - \frac{4}{\Delta t}{}^{t}U^{(k)} - {}^{t}\ddot{U} \tag{19} $$

将Eq.19带入到Eq.15中，可以得到Eq.20:

$$ \begin{aligned} \left({}^{t}K + \frac{4}{\Delta t^2}M\right)\Delta U^{(k)} &= {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(k-1)} \\ &\quad - M\left[\frac{4}{\Delta t^2}\left({}^{t+\Delta t}U^{(k-1)} - {}^{t}U\right) - \frac{4}{\Delta t}{}^{t}\dot{U} - {}^{t}\ddot{U}\right] \end{aligned} \tag{20} $$

算法流程如下：

1. 根据初始条件${}^{t+\Delta t}U^{(0)} = {}^{t}U$, ${}^{t+\Delta t}\dot{U}^{(0)} = {}^{t}U$, ${}^{t+\Delta t}\ddot{U}^{(0)} = {}^{t}U$, ${}^{t+\Delta t}K^{(0)} = {}^{t}K$, ${}^{t+\Delta t}F^{(0)} = {}^{t}F$，通过Eq.20计算$\Delta U^{(1)}$

2. 由Eq.16 计算${}^{t+\Delta t}U^{(1)}$

3. 由${}^{t+\Delta t}U^{(1)}$计算${}^{t+\Delta t}F^{(1)}$

4. 接下来通过Eq.20计算$\Delta U^{(2)}$重复步骤231，直到$\Delta U^{(k)}$足够小，结束循环。令${}^{t+\Delta t}U ={}^{t+\Delta t} U^{(k)}$

Explicit Central Difference Method

中心差分法不涉及变量β和γ，而是使用简单的中心差分方法：

$$ {}^{t}\ddot{U} = \frac{1}{\Delta t^2}\left({}^{t+\Delta t}U - 2{}^{t}U +{}^{t-\Delta t} U\right) \tag{21} $$

$$ {}^{t}\dot{U} = \frac{1}{2\Delta t}\left({}^{t+\Delta t}U -{}^{t-\Delta t} U\right) \tag{22} $$

将Eq.和Eq.带入Eq.23中，可以得到Eq.24。简记为Eq.

$$ M \cdot {}^{t}\ddot{U} + C \cdot {}^{t}\dot{U} + K \cdot {}^{t}U ={}^{t} R \tag{23} $$

$$ {}^{t+\Delta t}U\left(\frac{1}{\Delta t^2}M + \frac{1}{2\Delta t}C\right) ={}^{t} R-{}^{t}U\left(K - \frac{2}{\Delta t^2}\right) -{}^{t-\Delta t}U\left(\frac{1}{\Delta t^2}M - \frac{1}{2\Delta t}C\right) \tag{24} $$

$$ {}^{t+\Delta t}\hat{M} ={}^{t} \hat{R} \tag{25} $$

注意初始时刻的计算需要用到${}^{0-\Delta t}U$，我们利用初始条件计算${}^{0-\Delta t}U$，见Eq.。注意Central Difference Method这种方法有条件稳定，Δt不能太大，需要满足CFL稳定条件。

$$ {}^{0-\Delta t}U ={}^{0} U -{}^{0} \dot{U}\Delta t +{}^{0} \ddot{U}\frac{\Delta t^2}{2} \tag{26} $$