Understanding Schemes Part II - The road to Spec (Coordinate Rings and Morphisms)

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Last time, we estabslished the basics of classical algebraic geometry over algebraically closed fields, and motivated the duality between algebra and geometry via Hilbert's Nullstellenstaz, and got our first clues about how to generalise algebraic geometry via the correspondence of maximal ideals to points.

Before defining MaxSpec outright, let us first see that it can indeed generalise classical varieties (After all, that is what we are looking to do). It's worth noting at this point that we are currently only dealing with affine varieties, but that projective varieties, whose definition needs a bit more care, are immensely important to algebraic geometry. We will eventually meet these objects, in the world of schemes.

The polynomial ring $k[x_1, \dots, x_n]$ gives us a ring of algebraic functions on k^n, whose ideals correspond to points. Naturally, one may seek the analogue of this for a general algebraic variety.

Let $V = V(I)$ be an algebraic variety for $I \trianglelefteq k[x_1, \dots, x_n]$ a radical ideal. Since we are interested in the structure defined by polynomials, the functions we want to consider are those that arise from polynomial functions. We make the following definition:

Definition: The Coordinate Ring $k[V]$ of an algebraic variety V is defined as follows:

$$k[V] = \{f : V \to k | \, \exists F \in k[x_1, \dots, x_n], \, F|_V = f\}$$

So as functions, the elements of $k[V]$ are functions $V \to k$ which are restrictions of polynomial functions (Note that if we endow k with the Zariski topology, such functions are continuous, but not all continuous functions are of this form), so they can be given explicitly by polynomials.

Note that this presentation needn't be unique. Consider, for example, the y-axis in k^2, defined by $V = V(\langle x \rangle) = \{(0, y) : y \in k\} \subseteq k^2$, and the polynomials $F(x, y) = y^2 + x^2, G(x, y) = y^2 - x^2$, then $G|_V = F|_V$ as functions, since we are consider the set of points for which $x = 0$. In fact, these presentations needn't even be the same on V! Take $V(\langle x^2 - 1 \rangle) \subseteq k^2$, then $G(x, y) = x^2$ and $F(x, y) = 1$ satisfy $F|_V = G|_V$

There is a more algebraic description of the coordinate ring: There is a natural map $k[x_1, \dots, x_n] \to k[V]$

given via considering a polynomial f as a function, and restricting it to V. The kernel of this map is the set of polynomials which map to $0 \in k[V]$, i.e. as functions they vanish on V, so this is exactly $I(V) = \sqrt{I}$ (By the Nullstellensatz). By definition, this map is surjective, so the first isomorphism theorem gives

$$k[V] \cong k[x_1, \ldots, x_n]/I(V) \tag{1}$$

This allows us to think of elements of $k[V]$ as polynomials, modulo the equivalence relation $f \sim g$ iff $f - g$ vanishes on V (i.e. $f - g \in I(V)$), so if $f|_V = g|_V$ as functions. Now the correspondence theorem for rings in addition to the final corollary from last time gives a bijective correspondence between points of V and maximal ideals in $k[V]$ (Equivalently, maximal ideals in $k[x_1, \ldots, x_n]$ containing I). This extends our points-ideals correspondence further to any affine algebraic set.

The set of maximal ideals of $k[V]$ is again in correspondence with the points of V, so that $\text{MaxSpec}(k[V])$ should be constructed such that it is homeomorphic (Or isomorphic, for a correct notion of isomorphism) to V. This gives us the following useful perspective, which we will cling onto for dear life for the next while to make sense of all of this:

For a general ring, we would like to think of R as a "coordinate ring" of polynomial functions on our set of points $\text{MaxSpec}(R)$. We will continue exploring this perspective soon, but first we define our prototype of MaxSpec.

Definition: Let R be a ring. Define $\text{MaxSpec}(R)$ as the topological space whose underlying set is the set of maximal ideals of R, and whose **closed** sets are of the form

$$V(I) = \{M \in \text{MaxSpec}(R) : M \supseteq I\} \tag{2}$$

For an ideal I, i.e. the set of maximal ideals containing it.

Of course, we could have defined $V(S)$ for a random subset $S \subseteq R$ of elements, and noted that $V(S) = V(\langle S \rangle)$ (Even more simply than before). This fits in with our understanding of R as the coordinate ring:

For a polynomial $f \in k[x_1, \ldots, x_n]$, evaluating at a point $f(p)$ is the same as looking at the image of f under the canonical map

$$k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/(x_1 - p_1, \ldots, x_n - p_n) \cong k \tag{3}$$

Thus for a general ring, we can think of an element $r \in R$ as a function on $\text{MaxSpec}(R)$. In classical AG, since k is algebraically closed, every quotient by a maximal ideal is canonically isomorphic to to k, but generally this is not the case, so this function will have its range vary from point to point. In other words, for every $r \in R$ we a map

$$r : \mathrm{MaxSpec}(R) \to \coprod_{M \in \mathrm{MaxSpec}(R)} R/M,\ r(M) = r + M \tag{4}$$

Although the range varies, it still makes sense to define a 'vanishing set' as the set of maximal ideals in which $r(M) = 0$, or equivalently $r \in M$. This definition coincides with the definition we gave above, i.e. if $I \trianglelefteq R$ is an ideal, then we can define

$$V(I) = \{M \in \mathrm{MaxSpec}(R) : \forall r \in I, r(M) = 0\} = \{M \in \mathrm{MaxSpec}(R) : I \subseteq M\} \tag{5}$$

So now we have an operation $\mathrm{MaxSpec}$ which sends rings to spaces, and we have an inkling of intuition about how to think of this thing via the coordinate ring. The next step is to understand the types of maps we are interested in between these MaxSpecs. Obviously, any random continuous map won't do, since we are trying to do geometry, so we need extra structure. We must harken back to classical AG once again to try and find a generalisable notion of a map we are interested in.

For varities $V \subseteq k^n, W \subseteq k^m$, it is clear what the correct notion of function should be: Since we are dealing with structure coming from polynomials, similarly to the coordinate ring we shall define a Morphism to be a map $f : V \to W$ such that there are polynomial functions $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ for which $f = (f_1, \ldots, f_m)|_V$, where here we mean

$$(f_1, \ldots, f_m)(\bar{x}) = (f_1(x), \ldots, f_m(x)) \in k^m \tag{6}$$

Another reason these maps are natural is that it gives us another correspondence between algebra and geometry: Suppose we were looking for what the correct notion of a map of varieties is. Extending the analogy between algebra and geometry, we would want this notion to bijectively correspond to homomorphisms $\varphi : k[V] \to k[W]$. We are only considering k-Algebra homomorphisms, as this 'remembers' the underlying field k.

Given such a homomorphism φ, it is a fact that if $\mathfrak{m} \trianglelefteq k[W]$ is maximal, then $\varphi^{-1}(\mathfrak{m}) \trianglelefteq k[V]$ is maximal. Using the Nullstellensatz, this gives us a function $\varphi^{-1} : \mathrm{MaxSpec}(W) \to \mathrm{MaxSpec}(V)$, which in turn defines a function $f : W \to V$. Oops! We reversed the arrows! Indeed, the correspondence will be between 'nice' maps $V \to W$ and k-Algebra homomorphisms $k[W] \to k[V]$.

So now given a map $\varphi : k[W] \to k[V]$ we get an induced map $\varphi^{\#} : V \to W$, given by $\varphi(p) = q$, where $\varphi^{-1}(\mathfrak{m}_p) = \mathfrak{m}_q$. It turns out that this map $\varphi^{\#}$ is a morphism in the sense previously defined! So is given by polynomials. A morphism $f : V \to W$ also gives rise to a map $f^* : k[W] \to k[V]$, which is actually given by a precomposition with polynomials.

For the category theory inclined, we have defined a functor

$$\mathrm{F} : \mathrm{Var}_k \to (\text{Finitely generated, reduced } k\text{-algebras})^{op} \tag{7}$$

Where the morphisms are as defined previously. We restrict to finitely generated reduced k-Algebras because these are exactly the algebras given by a quotient of $k[x_1, \ldots, x_\ell]$ for any $\ell \geq 0$ by a radical ideal. This functor is actually an equivalence of categories, so Finitely generated reduced k-Algebras are *dual* to Affine Varieties.

Welcome back, everyone who skipped the last prargraph. The bottom line is that the notion of "is given by polynomials" is not easy to capture. We return to our perspective of R as the coordinate ring of $\mathrm{MaxSpec}(R)$. With this perspective in mind, we can take the 'Algebraic' analogue, i.e. a map of k-Algebras, or in the more general case, a map of rings $f : S \to R$, and try defining a morphism to be a map of the form $f^{-1} : \mathrm{MaxSpec}(R) \to \mathrm{MaxSpec}(S)$ for $f : S \to R$.

Unfortunately, for a general ring homomorphism $f^{-1}(M)$ is not necessarily a maximal ideal if $M \trianglelefteq R$ is. It *is* a prime ideal though. The idea to then keep track of prime ideals arises, in order to give an analogy to the notion of a variety morphism. One can think of this as retaining the information of irreducible subvarieties, since this is what prime ideals correspond to classicaly. This gives rise to the notion of Spec. Confusingly, points $\mathfrak{p} \in \mathrm{Spec}(R)$ don't have to be closed, even though they correspond to closed sets, i.e. $\{\mathfrak{p}\} \neq \overline{\{\mathfrak{p}\}}$. Another way to think of these points is as generic points of the subvariety corresponding to $\mathfrak{p}$. Note that in general, $V(\mathfrak{p})$ needn't be prime. This is already true in the classical notion if we move to a non-algebraically closed field ($\mathbb{R}[x, y]$ gives us an example). We keep track of this information about 'irreducibility' through the generic points $\mathfrak{p}$ ($\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$).

For all this and more, join us next time, hopefully soon.