Understanding Schemes Part I - Classical Algebraic Geometry

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To understand where the definitions of schemes even come from, we must first study classical algebraic geometry. The set-up is as follows:

Let k be a an algebraically closed field. This means that if $f \in k[x]$ is a polynomial, every root of f is already in k. This is an important condition for technical reasons (Intuitively, since we are interested in zero sets of polynomials, we do not want 'missing information' in the form of missing would-be vanishing points). Algebraic Geometry originated with the study of polynomials over the complex numbers.

We consider k^n, i.e. n-tuples of elements of k. We can think of multivariate polynomials $f \in k[x_1, \dots, x_n]$ as functions $f : k^n \to k$ by plugging in an n-tuple of elements of k into the variables of f.

We want to study sets which are "carved out" by multivariate polynomials. For example, the polynomial equation $x^2 + y^2 = 1$, or, rewritten $x^2 + y^2 - 1 = 0$ defines a circle over the real numbers (I know I said we are working with algebraically closed fields, but it is much easier to picture the real numbers).

More generally, we define an affine algebraic set (Also called an affine variety), which will be our objects of interest as follows:

Definition: Let $S \subseteq k[x_1, \dots, x_n]$ be a set of polynomials, define the set $V(S) \subseteq k^n$, the algebraic set defined by S via

$$\{x \in k^n : f(x) = 0, \forall f \in S\} \tag{1}$$

Note that for a point $x \in V(S)$, if $f, g \in S$ then $f(x) + g(x) = (f+g)(x) = 0$, and if $h \in k[x_1, \dots, x_n]$ then $h(x)f(x) = 0$. This closure under addition and absorption tells us that we can pass to ideals, i.e. if $\langle S \rangle$ is the ideal generated by S (i.e. the smallest ideal containing it) then $V(S) = V(\langle S \rangle)$. This will hopefully allow us to draw algebraic information from the geometric information of the algebraic set. We can refine this further, but we will wait until later to do so. For now, we define the Zariski Topology:

Definition: The Zariski Topology on k^n is defined by declaring the algebraic sets as the closed sets. Verify for yourself that the following holds:

1. If $I_1, \dots, I_n$ are ideals, then $\bigcup_{j=1}^{n} V(I_j) = V(\prod_{j=1}^{n} I_j)$ so these sets are

closed under finite unions.

2. If $\{I_j\}_j$ is a family of ideals, then $V(\sum_j I_j) = \bigcap_j V(I_j)$, so these sets are closed under aribtrary intersections.

3. $V(0) = k^n$ and $V(k[x_1, \ldots, x_n]) = \varnothing$ (Since no point, not even 0 vanishes on constant nonzero polynomials).

So the algebraic sets have the exact properties of closed sets in a topological space. If you've never seen this, it might be weird to think as a topology as being defined by its closed sets, but since closed and open sets are just complements of each other, declaring the closed sets is the same as declaring the open sets (And the properties they have to satisfy are dual).

Moreover, the Zariski Topology is quite weird. For example, it is almost never Hausdorff (In fact, a lot open sets are dense in closed sets! If X is closed and irreducible, i.e. not the union of two smaller closed sets, then every nomepty open subset of X in the subspace topology is dense, so every two nonempty open sets intersect nontrivially). Nonetheless, it is useful for reasoning about varieties, and among other things, gives the correct notion of 'dimension'.

We now give an open basis for the topology:

Let $0 \neq f \in k[x_1, \ldots, x_n]$, and let $D(f) = \{x \in k^n : f(x) \neq 0\}$ i.e. the set of all points on which f is nonzero. Note that this is the complement of $V(\langle f \rangle)$, i.e. the set of all points on which f is 0, hence it is open. Now, this is a basis for the Zariski topology, since if I is an ideal defining a closed set $V(I)$, then consider the union $\bigcup_{f \in I} D(f)$. Convnice yourselves that $V(I)^c = \bigcup_{f \in I} D(f)$, since a point isn't in the zero locus of I iff it doesn't vanish under one of the polynomials of I, so $D(f)$ are a basis for the Zariski Topology.

Note that we can also go the other way, i.e. we can talk about ideals that are determined by subsets of k^n:

Let $S \subseteq k^n$, then the ideal defined by S is

$$I(S) = \{f \in k[x_1, \ldots, x_n] : f(x) = 0, \forall x \in S\} \tag{2}$$

This is dual to the previous definition. Verify for yourselves that this is an ideal. Now the natural question is what is $V(I(S))$? It turns out that this is the smallest algebraic set containing S:

Theorem: $\overline{S} = V(I(S))$, i.e. the smallest closed set in the Zariski Topology containing S is $V(I(S))$.

Proof. An equivalent characterisation of the closure is the intersection of all closed sets containing S, hence by definition $\overline{S} \subseteq V(I(S)$. Converesly, let

$x \in V(I(S))$. We shall prove that x is in $\overline{S}$. Concretely, this means that every polynomial that vanishes on S vanishes on x as well (x cannot be 'separated' from S by polynomials).

Indeed, let f be a polynomial that vanishes on S, then $f \in I(S)$, but $x \in V(I(S))$, hence $f(x) = 0$, simple as that we have $V(I(S)) = \overline{S}$. $\square$

In particular, if we start with an algebraic set $X = V(J)$, then $V(I(X)) = X = V(I(V(J)))$. It is now natural to ask what is $I(V(J))$ for an ideal J, in order to be able to characterise the family of ideals for which ,$I(V(J)) = J$ (i.e. ideals coming from Algebraic subsets), and establish a correspondence between algebraic sets (Which we restrict to because they give $V(I(X)) = X$) and these ideals

$$\begin{aligned}
&\{\text{Algebraic Sets}\} = \{X \subseteq k^n : V(I(X)) = X\} \\
&\leftrightarrow \\
&\{I \trianglelefteq k[x_1, \dots, x_n] : I(V(I)) = I\} \\
&X \mapsto I(X),\, I \mapsto V(I)
\end{aligned}$$

Note that this correspondence is also order reversing, if $X \subseteq Y$ then $I(X) \supseteq I(Y)$ (It is easier to vanish on less points) and if $I \subseteq J$ then $V(J) \supseteq V(I)$ (It is easier to a vanishing point of less polynomials).

The previous theorem showed that V is a left inverse to I on algebraic sets. It turns out that we need to restrict our attention to a more specific type of ideal. In particular, suppose that $I(X)$ is the ideal of an algebraic set. Suppose that f is a polynomial such that $f^n \in I(X)$, then $f^n(x) = 0$ for all $x \in X$, but $f^n(x) = 0$ if and only if $f(x) = 0$ (Here we mean we are taking the polynomial to the n-th power), hence $f \in I(X)$ as well.

Such ideals are called <u>Radical Ideals</u>, and the wonderful Hilbert's Nullstellensatz tells us that this is the exact property that characterises the ideals of variaties. It is also important in giving us the motivation for schemes:

Theorem: (Hilbert's Nullstellensatz): Let $I \trianglelefteq k[x_1, \dots, x_n]$ be an ideal, then

$$I(V(I)) = \sqrt{I} = \{f : \exists n \in \mathbb{N},\, f^n \in I\} \tag{3}$$

Although the proof is quite nice, we skip it here for brevity, we are only skimming this story after all, and even so there is still so much to do and define! As a corollary, we get that there is a bijective correspondence given by V and I between algebraic sets and <u>Radical</u> ideals, since $\sqrt{I} = I$ for a radical ideal. Note that if $S \subset k^n$, then $I(S) = I(V(I(S)) = I(\overline{S})$ by the Nullstellensatz, since $I(S)$ is radical (the proof given for $I(X)$ for X an algebraic set works for a general subset S as well), so ideals do not distinguish between algebraic sets and their closures, just like algebraic sets don't distinguish between subsets and the (radical) ideals generated by them.

Moreover, it gives us the following important corollary:

Corollary: The maximal ideals of $k[x_1, \ldots, x_n]$ are in bijective correspondence to points $a = (a_1, \ldots, a_n) \in k^n$, in particular $\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$, and $a = V(\mathfrak{m}_a)$

Proof. First, let $a \in k^n$. Note that $x_1 - a_1, \ldots, x_n - a_n \in I(a)$, hence $(x_1 - a_1, \ldots, x_n - a_n) \subseteq I(a)$. Conversely, if $f \in I(a)$, then note that $f(a) = 0$, but this means that in the quotient $k[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n)$ we have $\bar{f}(x) = \bar{f}(a) = \overline{(f(a))} = 0$, so $f \in (x_1 - a_1, \ldots, x_n - a_n)$. It is possible to see this in more concrete ways as well: we can define $f(x_1 + a_1, \ldots, x_n + a_n) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$. Now we have

$$f(x_1, \cdots, x_n) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n} \tag{4}$$

Now if we just show that $c_{0,\ldots,0} = 0$, then every other term will be in $(x_1 - a_1, \ldots, x_n - a_n)$, hence we will have $f \in (x_1 - a_1, \ldots, x_n - a_n)$. Indeed, we have $c_{0,\ldots,0} = f(a_1, \ldots, a_n) = 0$ as all other terms vanish and $f \in I(a)$ by assumption, hence $f \in (x_1 - a_1, \ldots, x_n - a_n)$.

Now, suppose that we have $a, b \in k^n$ with $\mathfrak{m}_a = \mathfrak{m}_b$, then for every $1 \leq i \leq n$, we have $x_i - a_i \in \mathfrak{m}_b$, hence $b_i - a_i = 0$, so $a_i = b_i$ for all $1 \leq i \leq n$, hence $a = b$. This shows injectivity of the correspondence

Finally, we show surjectivity: Let $\mathfrak{m}$ be a maximal ideal, then $V(\mathfrak{m})$ is nonempty (By the Nullstellensatz, otherwise we would have $\mathfrak{m} = k[x_1, \ldots, x_n]$). Let $a \in V(\mathfrak{m})$, then $I(a) = \mathfrak{m}_a \supseteq I(V(\mathfrak{m})) = \mathfrak{m}$ (Here we are using the order-reversing property + the Nullstellensatz), hence by maximality $\mathfrak{m} = \mathfrak{m}_a$. This shows that the ideal $(x_1 - a_1, \ldots, x_n - a_n) = \mathfrak{m}_a$ is indeed maximal, completing the proof. □

This lays the ground for classical algebraic geometry, but keep in mind there is a whole world of theory that we have not discussed and will not be getting into. Just keep in mind that there are a lot of powerful theorems and techniques.

This correspondence between maximal ideals (purely algebraic) and points (geometric) gives a natural idea of how to generalise the powerful techniques of classical algebraic geometry and use them to study arbitrary commutative rings: Just substitute maximal ideals for points, and follow the geometry (Define a Zariski Topology on the sets of maximal ideals so that it acts as in the classical case, in particular it should make $k[x_1, \ldots, x_n]$ homeomorphic to k^n via the identification of maximal ideals and points). We shall explore this method and see if it holds up next time. In particular, we give some more motivation through coordinate rings.