Ring of fractions and localization

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We describe a construction known as the ring of fractions that has elements of the form a/b. It is also known as the ring of quotients, but it sounds too similar to the quotient ring, a completely different concept.

Definition 1. *A nonempty subset S of a ring R is multiplicative provided that $a, b \in S \implies ab \in S$.*

Theorem 1. *Let S be a multiplicative subset of a commutative ring R. The relation defined on the set $R \times S$ defined by*

$$(r, s) \sim (r', s') \iff \exists s_1 \in S, s_1(rs' - r's) = 0$$

is an equivalence relation. Furthermore, if R has no zero divisors and $0 \in R \setminus S$, then

$$(r, s) \sim (r', s') \iff rs' - r's = 0$$

Proof. $(r, s) \sim (r, s)$ and $(r, s) \sim (r', s') \iff (r', s') \sim (r, s)$ are obvious. Assume $(r, s) \sim (r', s'), (r', s') \sim (r'', s'')$. Then $\exists s_1, s_2 \in S$ such that $s_1(rs' - r's) = 0$ and $s_2(r's'' - r''s') = 0$. Then

$$s_1 s_2 s'(rs'' - r''s) = 0$$

Thus $(r, s) \sim (r'', s'')$. If R has no zero divisors and $0 \in R \setminus S$, then $\exists s_1 \in S, s_1(rs' - r's) = 0$. Since s_1 is not a zero divisor, $rs' - r's = 0$. □

Let S be a multiplicative subset of a commutative ring R and $\sim$ the equivalence relation from earlier. The equivalence class of $(r, s) \in R \times S$ is denoted r/s. The set of all equivalence classes is denoted $S^{-1}R$. Verify that

(i) $r/s = r'/s' \iff \exists s_1 \in S, s_1(rs' - r's) = 0$

(ii) $tr/ts = r/s$ for all $r \in R, t, s \in S$

(iii) If $0 \in S$, then $S^{-1}R$ consists of a single equivalence class.

Theorem 2. *Let S be a multiplicative subset of a commutative ring R and let $S^{-1}R$ be the set of equivalence classes under $(r, s) \sim (r', s') \iff \exists s_1 \in S, s_1(rs' - r's) = 0$.*

(i) $S^{-1}R$ is a commutative ring with identity, where addition and multiplication is defined by

$$r/s + r'/s' = (rs' + r's)/ss', \quad (r/s)(r'/s') = rr'/ss'$$

(ii) If R is a nonzero ring with no zero divisors and $0 \notin S$, $S^{-1}R$ is an integral domain.

(iii) If R is a nonzero ring with no zero divisors and S is the set of all nonzero elements of R, then $S^{-1}R$ is a field.

Proof. (i) If $r/s = r_1/s_1$, $r'/s' = r_1'/s_1'$, then $\exists s_2, s_3 \in S$ such that

$$s_2(rs_1 - r_1 s) = 0, \quad s_3(r's_1' - r_1's') = 0$$

Multiply the first equation by $s_3 s' s_1'$ and the second by $s_2 s s_1$. Add to get

$$s_2 s_3[(rs' + r's)s_1 s_1' - (r_1 s_1' + r_1' s_1)ss'] = 0$$

Therefore $(rs' + r's)/ss' = (r_1 s_1' + r_1' s_1)/s_1 s_1'$. If we instead multiply the first equation by $s_1' s_3 r'$ and the second by $s s_2 r_1$, then add, we get

$$s_2 s_3[rr' s_1 s_1' - r_1 r_1' s s'] = 0$$

Thus $rr'/ss' = r_1 r_1'/s_1 s_1'$. Hence addition and multiplication on $S^{-1}R$ is well-defined.
(ii). If R has no zero divisors and $0 \notin S$, then $r/s = 0/s \iff r = 0$. Thus $(r/s)(r'/s') = 0 \iff rr' = 0$. Since $rr' = 0$ iff $r = 0$ or $r' = 0$, it follows that $S^{-1}R$ is an integral domain.
(iii). If $r \neq 0$, the multiplicative inverse of r/s is s/r. □

The ring $S^{-1}R$ is called the ring of fractions of R by S. When R is an integral domain and S the set of all nonzero elements, $S^{-1}R$ is called the fraction field or quotient field of R. If R is a nonzero commutative ring and S is the set of all elements of R that are not zero divisors of R, then $S^{-1}R$ is called the total ring of fractions of R.

Theorem 3. *Let S be a multiplicative subset of a commutative ring R.*

(i) The map $\phi_S : R \to S^{-1}R$ given by $r \mapsto rs/s$ for any $s \in S$ is a well-defined homomorphism of rings such that $\phi_S(s)$ is a unit in $S^{-1}R$ for every $s \in S$.

(ii) If $0 \notin S$ and S contains no zero divisors, then ϕ_S is a monomorphism. In particular, any integral domain may be embedded in its quotient field.

(iii) If R has an identity and S consists of units, then ϕ_S is an isomorphism. In particular, the total ring of fractions of a field F is isomorphic to F.

Proof. (i). If $s, s' \in S$, then $rs/s = rs'/s'$ whence ϕ_S is well-defined. Verify that ϕ_S is a ring homomorphism and that $\forall s \in S, s/s^2$ is the multiplicative inverse of $s^2/s = \phi_S(s)$.
(ii). If $\phi_S(r) = 0, \exists s_1 \in S, rs^2 s_1 = 0$. Since $s^2 s_1 \neq 0$, $r = 0$.
(iii). ϕ_S is a monomorphism by (ii). If $r/s \in S^{-1}R$, $r/s = \phi_S(rs^{-1})$. Thus ϕ_S is an isomorphism. □

It is customary to identify an integral domain R with its image under ϕ_S and consider R as a subring of its quotient field.

Theorem 4. *Let S be a multiplicative subset of a commutative ring R and let T be any commutative ring with identity. If $f : R \to T$ is a homomorphism of rings such that $f(s)$ is a unit in T for all $s \in S$, then there exists a unique homomorphism of rings $\bar{f} : S^{-1}R \to T$ such that $\bar{f} \circ \phi_S = f$. The ring $S^{-1}R$ is completely determined up to isomorphism by this property.*

Proof. Verify that $\bar{f} : S^{-1}R \to T$ given by $\bar{f}(r/s) = f(r)f(s)^{-1}$ is a well-defined homomorphism of rings such that $\bar{f} \circ \phi_S = f$. If $g : S^{-1}R \to T$ is another homomorphism such that $g \circ \phi_S = f$, then $\forall s \in S, g(\phi_S(s))$ is a unit in T. $g(\phi_S(s)^{-1}) = g(\phi_S(s))^{-1}$.

$$g(r/s) = g(\phi_S(r)\phi_S(s)^{-1}) = f(r)f(s)^{-1} = \bar{f}(r/s)$$

Thus $g = \bar{f}$. Let $\mathscr{C}$ be the category whose objects are all (f, T) where T is a commutative ring with identity and $f : R \to T$ a homomorphism of rings such that $f(s)$ is a unit in T for every $s \in S$. Define a morphism in $\mathscr{C}$ from (f_1, T_1) to (f_2, T_2) to be a homomorphism of rings $g : T_1 \to T_2$ such that $g \circ f_1 = f_2$. Verify that $\mathscr{C}$ is a category and that a morphism $g : (f_1, T_1) \to (f_2, T_2)$ in $\mathscr{C}$ is an equivalence iff $g : T_1 \to T_2$ is an isomorphism of rings. Then by the preceding work we have shown $(\phi_S, S^{-1}R)$ is a universal object in $\mathscr{C}$, whence $S^{-1}R$ is completely determined up to isomorphism. □

Corollary 1. *Let R be an integral domain considered as a subring of its quotient field F. If E is a field and $f : R \to E$ a monomorphism of rings, then there is a unique monomorphism of fields $\bar{f} : F \to E$ such that $\bar{f}|_R = f$. In particular, any field E_1 containing R contains an isomorphic copy F_1 of F with $R \subseteq F_1 \subseteq E_1$.*

Proof. Let $S = R \setminus \{0\}$ and apply the theorem to $f : R \to E$ to obtain a homomorphism $\bar{f} : F \to E$ such that $\bar{f} \circ \phi_S = f$. $\bar{f}$ is a monomorphism. Since R is identified with $\phi_S(R)$, $\bar{f}|_R = f$. Last statement follows when $f : R \to E_1$ is the inclusion map. □

Theorem 5. *Let S be a multiplicative subset of a commutative ring R.*

(i) If I is an ideal in R, then $S^{-1}I$ is an ideal in $S^{-1}R$.

(ii) If J is another ideal in R, then

$$S^{-1}(I + J) = S^{-1}I + S^{-1}J$$

$$S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$$

$$S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J$$

Proof. Note that $\sum_{i=1}^{n} c_i/s = (\sum_{i=1}^{n} c_i)/s$, $\sum_{j=1}^{m} a_j b_j/s = \sum_{j=1}^{m}(a_j/s)(b_j s/s)$

$$\sum_{k=1}^{t} c_k/s_k = (\sum_{k=1}^{t} c_k s_1 s_2 \cdots s_{k-1} s_{k+1} \cdots s_t)/s_1 s_2 \cdots s_t$$

Let $a/s = b/s \in S^{-1}I \cap S^{-1}J$ where $a \in I, b \in J$. Then $\exists s' \in S, s'a = s'b \in I \cap J$.

$$a/s = s'a/s's = s'b/s's = b/s$$

Thus $a/s \in S^{-1}(I \cap J)$. □

Theorem 6. *Let S be a multiplicative subset of a commutative ring R with identity and let I be an ideal of R. Then $S^{-1}I = S^{-1}R$ iff $S \cap I \neq \emptyset$.*

Proof. If $s \in S \cap I$, $1 = s/s \in S^{-1}I$ and hence $S^{-1}I = S^{-1}R$. Conversely, if $S^{-1}I = S^{-1}R$, $\phi_S^{-1}(S^{-1}I) = R$ whence $\exists a \in I, s \in S, \phi_S(1) = a/s$. Since $\phi_S(1) = 1s/s$, $\exists s_1 \in S, s^2 s_1 = ass_1$. Thus $S \cap I \neq \emptyset$. □

$S^{-1}I$ is called the extension of I in $S^{-1}R$. If J is an ideal in $S^{-1}R$, then $\phi_S^{-1}(J)$ is an ideal in R. $\phi_S^{-1}(J)$ is called the contraction of J in R.

Lemma 1. *Let S be a multiplicative subset of a commutative ring R with identity and let I be an ideal in R.*

(i) $I \subseteq \phi_S^{-1}(S^{-1}I)$

(ii) If $I = \phi_S^{-1}(J)$ for some ideal J in $S^{-1}R$, then $S^{-1}I = J$. Every ideal in $S^{-1}R$ is of the form $S^{-1}I$ for some ideal I in R.

(iii) If P is a prime ideal in R and $S \cap P = \emptyset$, then $S^{-1}P$ is a prime ideal in $S^{-1}R$ and $\phi_S^{-1}(S^{-1}P) = P$.

Proof. (i). If $a \in I, \forall s \in S, as \in I$. Thus $\phi_S(a) = as/s \in S^{-1}I$ whence $a \in \phi_S^{-1}(S^{-1}I)$.
(ii). Since $I = \phi_S^{-1}(J)$, every element of $S^{-1}I$ is of the form r/s where $\phi_S(r) \in J$. Thus $r/s = (1_R/s)(rs/s) \in J$ whence $S^{-1}I \subseteq J$. Conversely, if $r/s \in J$, then $\phi_S(r) = rs/s = (r/s)(s^2/s) \in J$ whence $r \in \phi_S^{-1}(J) = I$. Thus $r/s \in S^{-1}I$ and $J \subseteq S^{-1}I$.
(iii). $S^{-1}P$ is an ideal such that $S^{-1}P \neq S^{-1}R$ by the previous theorem. If $(r/s)(r'/s') \in S^{-1}P$ then $rr'/ss' = a/t$ with $a \in P, t \in S$. $\exists s_1 \in S, s_1 trr' = s_1 ass' \in P$. Since $s_1 t \in S$ and $S \cap P = \emptyset$, $rr' \in P$ whence $r \in P$ or $r' \in P$. Thus $r/s \in S^{-1}P$ or $r'/s' \in S^{-1}P$. Thus $S^{-1}P$ is prime. $P \subseteq \phi_S^{-1}(S^{-1}P)$ by (i). If $r \in \phi_S^{-1}(S^{-1}P)$, then $\phi_S(r) \in S^{-1}P$. Thus $rs/s = a/t$ with $a \in P, t \in S$. $\exists s_1 \in S$, $s_1 str = s_1 sa \in P$. Since $s_1 st \in S, S \cap P = \emptyset, r \in P$. □

Theorem 7. *Let S be a multiplicative subset of a commutative ring R with identity. Then there is a one-to-one correspondence between the set $\mathfrak{U}$ of prime ideals of R which are disjoint from S and the set $\mathfrak{V}$ of prime ideals of $S^{-1}R$, given by $P \mapsto S^{-1}P$.*

Proof. By the preceding lemma, $\mathfrak{U} \to \mathfrak{V} : P \mapsto S^{-1}P$ is injective. Let J be a prime ideal of $S^{-1}R$ and let $P = \phi_S^{-1}(J)$. Since $S^{-1}P = J$, it suffices to show that P is prime. If $ab \in P, \phi_S(a)\phi_S(b) = \phi_S(ab) \in J$. Since J is prime, $\phi_S(a) \in J$ or $\phi_S(b) \in J$. Thus either $a \in P$ or $b \in P$. □

Let R be a commutative ring with identity and P a prime ideal of R. Then $S = R \setminus P$ is multiplicative. $S^{-1}R$ is called the localization of R at P and is denoted R_P. If I is an ideal in R, then the ideal $S^{-1}I$ in R_P is denoted I_P.

Theorem 8. *Let P be a prime ideal in a commutative ring R with identity.*

(i) There is a one-to-one correspondence between the set of prime ideals of R contained in P and the set of prime ideals of R_P given by $Q \mapsto Q_P$.

(ii) The ideal P_P in R_P is the unique maximal ideal of R_P.

Proof. The prime ideals contained in P are precisely those disjoint from $S = R \setminus P$. (i) follows from the previous theorem. If M is a maximal ideal of R_P, then M is prime. Whence $M = Q_P$ for some prime ideal Q of R with $Q \subseteq P$. But $Q \subseteq P$ implies $Q_P \subseteq P_P$. Since $P_P \neq R_P$, we must have $Q_P = P_P$. Thus P_P is the unique maximal ideal in R_P. □

Definition 2. *A local ring is a commutative ring with identity which has a unique maximal ideal.*

Theorem 9. *If R is a commutative ring with identity then the following conditions are equivalent*

(i) R is a local ring

(ii) All nonunits of R are contained in some ideal $M \neq R$

(iii) The nonunits of R form an ideal

Proof. If I is an ideal of R and $a \in I$, then $(a) \subseteq I$. Thus $I \neq R$ iff I consists only of nonunits. These facts imply (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i).
(i) $\Longrightarrow$ (ii). If $a \in R$ is a nonunit, then $(a) \neq R$. Thus (a) is contained in the unique maximal ideal of R. □