Jordan-Hölder Theorem

written by Night Shift in Math on Functor Network
original link: https://functor.network/user/854/entry/380

Definition 1. *A subnormal series of a group G is a chain of subgroups $G = G_0 \geq G_1 \geq \cdots \geq G_n$ such that $G_{i+1} \trianglelefteq G_i$ for $i \in \{0, 1, \cdots, n\}$. The factors of the series are the quotient groups G_i/G_{i+1}. The length of the series is the number of strict inclusions. A subnormal series such that $G_i \trianglelefteq G$ for all i is said to be a normal series.*

As an example, the derived series $G \geq G^{(1)} \geq \cdots \geq G^{(n)}$ is a normal series for any group G. If G is nilpotent, the ascending central series $C_1(G) \leq C_2(G) \leq \cdots \leq C_n(G) = G$ is a normal series for G.

Definition 2. *Let $G = G_0 \geq G_1 \geq \cdots \geq G_n$ be a subnormal series. A one-step refinement of this series is any series of the form $G = G_0 \geq \cdots \geq G_i \geq N \geq G_{i+1} \geq \cdots \geq G_n$ or $G = G_0 \geq \cdots \geq G_n \geq N$ where $N \trianglelefteq G_i$ and $G_{i+1} \trianglelefteq N$ for $i < n$. A refinement of a subnormal series S is any subnormal series obtained from S by a finite sequence of one-step refinements. A refinement is said to be proper iff its length is larger than the length of S.*

Definition 3. *A subnormal series $G = G_0 \geq G_1 \geq \cdots \geq G_n = \langle e \rangle$ is a composition series iff each factor G_i/G_{i+1} is simple. A subnormal series $G = G_0 \geq G_1 \geq \cdots \geq G_n = \langle e \rangle$ is a solvable series iff each factor is abelian.*

A commonly used fact for composition series: When $G \neq N$, G/N is simple iff N is maximal in the set of all normal subgroups M of G with $M \neq G$.

Theorem 1. *(i) Every finite group G has a composition series.*

(ii) Every refinement of a solvable series is a solvable series.

(iii) A subnormal series is a composition series iff it has no proper refinements.

Proof. (i) Let G_1 be a maximal normal subgroup of G. Then G/G_1 is simple. Let G_2 be a maximal normal subgroup of G_1 and so on. Since G is finite, this process must end with $G_n = \langle e \rangle$. Thus $G > G_1 > G_2 > \cdots > G_n = \langle e \rangle$ is a composition series.

(ii) If G_i/G_{i+1} is abelian, $G_{i+1} \trianglelefteq H \trianglelefteq G_i$, then H/G_{i+1} is abelian since it is a subgroup of G_i/G_{i+1} and G_i/H is abelian since it is isomorphic to $(G_i/G_{i+1})/(H/G_{i+1})$.

(iii) If $G_{i+1} \triangleleft H \triangleleft G_i$, H/G_{i+1} is a proper normal subgroup of G_i/G_{i+1}. All proper normal subgroups of G_i/G_{i+1} have the form H/G_{i+1} for some $G_{i+1} \triangleleft H \triangleleft G_i$. □

Theorem 2. *A group G is solvable iff it has a solvable series.*

Proof. If G is solvable, $G \geq G^{(1)} \geq G^{(2)} \geq \cdots \geq G^{(n)} = \langle e \rangle$ is a solvable series. If $G \geq G_1 \geq G_2 \geq \cdots \geq G_n = \langle e \rangle$ is a solvable series for G, then G/G_1 abelian implies that $G_1 \geq G^{(1)}$. G_1/G_2 abelian implies $G_2 \geq G_1' \geq G^{(2)}$. Proceeding by induction conclude $\forall i, G_i \geq G^{(i)}$. In particular, $\langle e \rangle = G_n \geq G^{(n)}$ so G is solvable. □

Proposition 1. *A finite group G is solvable iff G has a composition series whose factors are cyclic of prime order.*

Proof. A composition series with cyclic factors is a solvable series. Conversely, assume $G \geq G_1 \geq G_2 \geq \cdots \geq G_n = \langle e \rangle$ is a solvable series for G. If $G_0 \neq G_1$, let H_1 be a maximal normal subgroup of G containing G_1. If $H_1 \neq G_1$, let H_2 be a maximal normal subgroup of G containing G_1. Continue until we obtain a series $G > H_1 > H_2 > \cdots > H_k > G_1$ with each subgroup a maximal normal subgroup of the preceding, whence each factor is simple. This series terminates as in, eventually $H_{k+1} = G_1$ since G is finite. Doing this for each pair (G_i, G_{i+1}) gives a solvable refinement $G = N_0 > N_1 > \cdots > N_r = \langle e \rangle$ of the original series. Each factor of this series is abelian and simple hence cyclic of prime order. □

Definition 4. *Two subnormal series S and T of a group G are equivalent iff there is a one-to-one correspondence between the nontrivial factors of S and the nontrivial factors of T such that corresponding factors are isomorphic groups.*

Lemma 1. *If S is a composition series of a group G, then any refinement of S is equivalent to S.*

Proof. Since S is a composition series, S has no proper refinements. Thus any refinement of S is obtained by inserting additional copies of G_i. Any refinement of S has the exact same nontrivial factors as S and is thus equivalent to S. □

Lemma 2 (Zassenhaus)**.** *Let A^*, A, B^*, B be subgroups of a group G such that $A^* \trianglelefteq A, B^* \trianglelefteq B$.*

(i) $A^*(A \cap B^*) \trianglelefteq A^*(A \cap B)$

(ii) $B^*(A^* \cap B) \trianglelefteq B^*(A \cap B)$

(iii) $A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$

Proof. Note $A \cap B^* = (A \cap B) \cap B^* \trianglelefteq A \cap B$. Similarly, $A^* \cap B \trianglelefteq A \cap B$. Thus $D = (A^* \cap B)(A \cap B^*) \trianglelefteq A \cap B$. Also, $A^*(A \cap B) \leq A, B^*(A \cap B) \leq B$. We will define an epimorphism $f : A^*(A \cap B) \to (A \cap B)/D$ with kernel $A^*(A \cap B^*)$. This would imply that $A^*(A \cap B^*) \trianglelefteq A^*(A \cap B)$ and that $A^*(A \cap B)/A^*(A \cap B^*) \cong (A \cap B)/D$. Define $f : A^*(A \cap B) \to (A \cap B)/D$ as follows: If $a \in A^*, c \in A \cap B$, let $f(ac) = Dc$. $ac = a_1c_1$ implies $c_1c^{-1} = a_1^{-1}a \in (A \cap B) \cap A^* = A^* \cap B \leq D$. Thus f is well defined. f is clearly surjective.

$$f((a_1c_1)(a_2c_2)) = f(a_1a_3c_1c_2) = Dc_1c_2 = Dc_1Dc_2 = f(a_1c_1)f(a_2c_2)$$

Finally, $ac \in \ker f \iff c \in D \iff c = a_1c_1, a_1 \in A^* \cap B, c_1 \in A \cap B^*$. Hence $ac \in \ker f \iff ac = (aa_1)c_1 \in A^*(A \cap B^*)$. $\ker f = A^*(A \cap B^*)$. A symmetric argument shows (ii) and $B^*(A \cap B)/B^*(A^* \cap B) \cong (A \cap B)/D$ whence (iii) follows. □

Theorem 3 (Schreier Refinement)**.** *Any two subnormal (resp. normal) series of a group G have subnormal (resp. normal) refinements that are equivalent.*

Proof. Let $G = G_0 \geq G_1 \geq \cdots \geq G_n$ and $G = H_0 \geq H_1 \geq \cdots \geq H_m$ be subnormal [resp. normal] series. Let $G_{n+1} = H_{m+1} = \langle e \rangle$ and for each $0 \leq i \leq n$, consider the groups

$$G_i = G_{i+1}(G_i \cap H_0) \geq G_{i+1}(G_i \cap H_1) \geq \cdots \geq G_{i+1}(G_i \cap H_m) \geq G_{i+1}(G_i \cap H_{m+1}) = G_{i+1}$$

For each $0 \leq j \leq m$, the Zassenhaus lemma applied to $G_{i+1}, G_i, H_{j+1}, H_j$ shows that $G_{i+1}(G_i \cap H_{j+1}) \trianglelefteq G_{i+1}(G_i \cap H_j)$ (if the original series is normal, each $G_{i+1}(G_i \cap H_{j+1}) \trianglelefteq G$) Inserting these groups between each G_i and G_{i+1}, denoting $G_{i+1}(G_i \cap H_j)$ by $G(i,j)$ gives a subnormal (resp. normal) refinement of $G_0 \geq G_1 \geq \cdots \geq G_n$. A symmetric argument shows there is a refinement of $H_0 \geq H_1 \geq \cdots \geq H_m$ with $H(i,j) = H_{j+1}(G_i \cap H_j)$.

$$G = G(0,0) \geq G(0,1) \geq \cdots \geq G(0,m) \geq G(1,0) \geq G(1,1) \geq \cdots \geq G(1,m) \geq G(2,0) \geq \cdots \geq G(n,m)$$

$$G = H(0,0) \geq H(1,0) \geq \cdots \geq H(n,0) \geq H(0,1) \geq H(1,1) \geq \cdots \geq H(n,1) \geq H(0,2) \geq \cdots \geq H(n,m)$$

By the Zassenhaus lemma, $G(i,j)/G(i,j+1) \cong H(i,j)/H(i+1,j)$. □

Theorem 4 (Jordan-Hölder)**.** *Any two composition series of a group G are equivalent. Therefore every group having a composition series determines a unique (up to permutation and isomorphism) list of simple groups.*

Proof. Since composition series are subnormal series, any two composition series have equivalent refinements by the Schreier Refinement Theorem. But every refinement of a composition series is equivalent to the original composition series. Thus any two composition series are equivalent. □