

# Jordan-Hölder Theorem

Night Shift in Math • 18 May 2024

**Definition 1.** A subnormal series of a group  $G$  is a chain of subgroups  $G = G_0 \geq G_1 \geq \cdots \geq G_n$  such that  $G_{i+1} \trianglelefteq G_i$  for  $i \in \{0, 1, \dots, n\}$ . The factors of the series are the quotient groups  $G_i/G_{i+1}$ . The length of the series is the number of strict inclusions. A subnormal series such that  $G_i \trianglelefteq G$  for all  $i$  is said to be a normal series.

As an example, the derived series  $G \geq G^{(1)} \geq \cdots \geq G^{(n)}$  is a normal series for any group  $G$ . If  $G$  is nilpotent, the ascending central series  $C_1(G) \leq C_2(G) \leq \cdots \leq C_n(G) = G$  is a normal series for  $G$ .

**Definition 2.** Let  $G = G_0 \geq G_1 \geq \cdots \geq G_n$  be a subnormal series. A one-step refinement of this series is any series of the form  $G = G_0 \geq \cdots \geq G_i \geq N \geq G_{i+1} \geq \cdots \geq G_n$  or  $G = G_0 \geq \cdots \geq G_n \geq N$  where  $N \trianglelefteq G_i$  and  $G_{i+1} \trianglelefteq N$  for  $i < n$ . A refinement of a subnormal series  $S$  is any subnormal series obtained from  $S$  by a finite sequence of one-step refinements. A refinement is said to be proper iff its length is larger than the length of  $S$ .

**Definition 3.** A subnormal series  $G = G_0 \geq G_1 \geq \cdots \geq G_n = \langle e \rangle$  is a composition series iff each factor  $G_i/G_{i+1}$  is simple. A subnormal series  $G = G_0 \geq G_1 \geq \cdots \geq G_n = \langle e \rangle$  is a solvable series iff each factor is abelian.

A commonly used fact for composition series: When  $G \neq N$ ,  $G/N$  is simple iff  $N$  is maximal in the set of all normal subgroups  $M$  of  $G$  with  $M \neq G$ .

**Theorem 1.** (i) Every finite group  $G$  has a composition series.

(ii) Every refinement of a solvable series is a solvable series.

(iii) A subnormal series is a composition series iff it has no proper refinements.

*Proof.* (i) Let  $G_1$  be a maximal normal subgroup of  $G$ . Then  $G/G_1$  is simple. Let  $G_2$  be a maximal normal subgroup of  $G_1$  and so on. Since  $G$  is finite, this process must end with  $G_n = \langle e \rangle$ . Thus  $G > G_1 > G_2 > \cdots > G_n = \langle e \rangle$  is a composition series.

(ii) If  $G_i/G_{i+1}$  is abelian,  $G_{i+1} \trianglelefteq H \trianglelefteq G_i$ , then  $H/G_{i+1}$  is abelian since it is a subgroup of  $G_i/G_{i+1}$  and  $G_i/H$  is abelian since it is isomorphic to  $(G_i/G_{i+1})/(H/G_{i+1})$ .

(iii) If  $G_{i+1} \triangleleft H \triangleleft G_i$ ,  $H/G_{i+1}$  is a proper normal subgroup of  $G_i/G_{i+1}$ . All proper normal subgroups of  $G_i/G_{i+1}$  have the form  $H/G_{i+1}$  for some  $G_{i+1} \triangleleft H \triangleleft G_i$ .  $\square$

**Theorem 2.** *A group  $G$  is solvable iff it has a solvable series.*

*Proof.* If  $G$  is solvable,  $G \geq G^{(1)} \geq G^{(2)} \geq \dots \geq G^{(n)} = \langle e \rangle$  is a solvable series. If  $G \geq G_1 \geq G_2 \geq \dots \geq G_n = \langle e \rangle$  is a solvable series for  $G$ , then  $G/G_1$  abelian implies that  $G_1 \geq G^{(1)}$ .  $G_1/G_2$  abelian implies  $G_2 \geq G_1' \geq G^{(2)}$ . Proceeding by induction conclude  $\forall i, G_i \geq G^{(i)}$ . In particular,  $\langle e \rangle = G_n \geq G^{(n)}$  so  $G$  is solvable.  $\square$

**Proposition 1.** *A finite group  $G$  is solvable iff  $G$  has a composition series whose factors are cyclic of prime order.*

*Proof.* A composition series with cyclic factors is a solvable series. Conversely, assume  $G \geq G_1 \geq G_2 \geq \dots \geq G_n = \langle e \rangle$  is a solvable series for  $G$ . If  $G_0 \neq G_1$ , let  $H_1$  be a maximal normal subgroup of  $G$  containing  $G_1$ . If  $H_1 \neq G_1$ , let  $H_2$  be a maximal normal subgroup of  $G$  containing  $G_1$ . Continue until we obtain a series  $G > H_1 > H_2 > \dots > H_k > G_1$  with each subgroup a maximal normal subgroup of the preceding, whence each factor is simple. This series terminates as in, eventually  $H_{k+1} = G_1$  since  $G$  is finite. Doing this for each pair  $(G_i, G_{i+1})$  gives a solvable refinement  $G = N_0 > N_1 > \dots > N_r = \langle e \rangle$  of the original series. Each factor of this series is abelian and simple hence cyclic of prime order.  $\square$

**Definition 4.** *Two subnormal series  $S$  and  $T$  of a group  $G$  are equivalent iff there is a one-to-one correspondence between the nontrivial factors of  $S$  and the nontrivial factors of  $T$  such that corresponding factors are isomorphic groups.*

**Lemma 1.** *If  $S$  is a composition series of a group  $G$ , then any refinement of  $S$  is equivalent to  $S$ .*

*Proof.* Since  $S$  is a composition series,  $S$  has no proper refinements. Thus any refinement of  $S$  is obtained by inserting additional copies of  $G_i$ . Any refinement of  $S$  has the exact same nontrivial factors as  $S$  and is thus equivalent to  $S$ .  $\square$

**Lemma 2** (Zassenhaus). *Let  $A^*, A, B^*, B$  be subgroups of a group  $G$  such that  $A^* \trianglelefteq A, B^* \trianglelefteq B$ .*

$$(i) \quad A^*(A \cap B^*) \trianglelefteq A^*(A \cap B)$$

$$(ii) \quad B^*(A^* \cap B) \trianglelefteq B^*(A \cap B)$$

$$(iii) \quad A^*(A \cap B)/A^*(A \cap B^*) \cong B^*(A \cap B)/B^*(A^* \cap B)$$

*Proof.* Note  $A \cap B^* = (A \cap B) \cap B^* \trianglelefteq A \cap B$ . Similarly,  $A^* \cap B \trianglelefteq A \cap B$ .

Thus  $D = (A^* \cap B)(A \cap B^*) \trianglelefteq A \cap B$ . Also,

$A^*(A \cap B) \leq A, B^*(A \cap B) \leq B$ . We will define an epimorphism

$f : A^*(A \cap B) \rightarrow (A \cap B)/D$  with kernel  $A^*(A \cap B^*)$ . This would imply that  $A^*(A \cap B^*) \trianglelefteq A^*(A \cap B)$  and that  $A^*(A \cap B)/A^*(A \cap B^*) \cong (A \cap B)/D$ .

Define  $f : A^*(A \cap B) \rightarrow (A \cap B)/D$  as follows: If  $a \in A^*, c \in A \cap B$ , let

$$f(ac) = Dc. \quad ac = a_1c_1 \text{ implies } c_1c^{-1} = a_1^{-1}a \in (A \cap B) \cap A^* = A^* \cap B \leq D.$$

Thus  $f$  is well defined.  $f$  is clearly surjective.

$$f((a_1c_1)(a_2c_2)) = f(a_1a_3c_1c_2) = Dc_1c_2 = Dc_1Dc_2 = f(a_1c_1)f(a_2c_2)$$

Finally,  $ac \in \ker f \iff c \in D \iff c = a_1c_1, a_1 \in A^* \cap B, c_1 \in A \cap B^*$ . Hence  $ac \in \ker f \iff ac = (aa_1)c_1 \in A^*(A \cap B^*)$ .  $\ker f = A^*(A \cap B^*)$ . A symmetric argument shows (ii) and  $B^*(A \cap B)/B^*(A^* \cap B) \cong (A \cap B)/D$  whence (iii) follows.  $\square$

**Theorem 3** (Schreier Refinement). *Any two subnormal (resp. normal) series of a group  $G$  have subnormal (resp. normal) refinements that are equivalent.*

*Proof.* Let  $G = G_0 \geq G_1 \geq \dots \geq G_n$  and  $G = H_0 \geq H_1 \geq \dots \geq H_m$  be subnormal [resp. normal] series. Let  $G_{n+1} = H_{m+1} = \langle e \rangle$  and for each  $0 \leq i \leq n$ , consider the groups

$$G_i = G_{i+1}(G_i \cap H_0) \geq G_{i+1}(G_i \cap H_1) \geq \dots \geq G_{i+1}(G_i \cap H_m) \geq G_{i+1}(G_i \cap H_{m+1})$$

For each  $0 \leq j \leq m$ , the Zassenhaus lemma applied to  $G_{i+1}, G_i, H_{j+1}, H_j$  shows that  $G_{i+1}(G_i \cap H_{j+1}) \trianglelefteq G_{i+1}(G_i \cap H_j)$  (if the original series is normal, each  $G_{i+1}(G_i \cap H_{j+1}) \trianglelefteq G$ ) Inserting these groups between each  $G_i$  and  $G_{i+1}$ , denoting  $G_{i+1}(G_i \cap H_j)$  by  $G(i, j)$  gives a subnormal (resp. normal) refinement of  $G_0 \geq G_1 \geq \dots \geq G_n$ . A symmetric argument shows there is a refinement of  $H_0 \geq H_1 \geq \dots \geq H_m$  with  $H(i, j) = H_{j+1}(G_i \cap H_j)$ .

$$G = G(0, 0) \geq G(0, 1) \geq \dots \geq G(0, m) \geq G(1, 0) \geq G(1, 1) \geq \dots \geq G(1, m)$$

$$G = H(0, 0) \geq H(1, 0) \geq \dots \geq H(n, 0) \geq H(0, 1) \geq H(1, 1) \geq \dots \geq H(n, 1)$$

By the Zassenhaus lemma,  $G(i, j)/G(i, j+1) \cong H(i, j)/H(i+1, j)$ .  $\square$

**Theorem 4** (Jordan-Hölder). *Any two composition series of a group  $G$  are equivalent. Therefore every group having a composition series determines a unique (up to permutation and isomorphism) list of simple groups.*

*Proof.* Since composition series are subnormal series, any two composition series have equivalent refinements by the Schreier Refinement Theorem. But every refinement of a composition series is equivalent to the original composition series. Thus any two composition series are equivalent.  $\square$