# Finitely generated abelian groups

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Finitely generated abelian groups have a structure theorem, which makes them easy to describe and enumerate completely. The structure theorem states that every finitely generated abelian group is a direct sum of $\mathbb{Z}$ and $\mathbb{Z}_{p^k}$ where $p$ is a prime.

**Definition 1.** *A group $G$ is indecomposable iff $G \neq \langle e \rangle$ and $G$ is not the internal direct product of two of its proper subgroups.*

**Lemma 1.** *$\mathbb{Z}$ and $\mathbb{Z}_{p^k}$ for $p$ a prime are indecomposable.*

*Proof.* Every nontrivial subgroup of $\mathbb{Z}$ is cyclic. Any two nontrivial subgroups $\langle n \rangle, \langle m \rangle$ have a non-trivial intersection at $(n, m)$. Thus $\mathbb{Z}$ cannot be a direct sum of those subgroups, hence $\mathbb{Z}$ is indecomposable. Suppose $\mathbb{Z}_{p^n} = A \oplus B$ is a nontrivial decomposition with $|A| = p^a$ and $|B| = p^b$ with $0 < a, b < n$. Then $p^{\max(a,b)}\mathbb{Z}_{p^n} = p^{\max(a,b)}A \oplus p^{\max(a,b)}B = 0$ which is a contradiction since $\mathbb{Z}_{p^n}$ has an element of order $p^n > p^{\max(a,b)}$. □

In other words, the structure theorem says that every finitely generated abelian group is the direct sum of a finite number of indecomposable groups.

**Lemma 2.** *$\mathbb{Z}_r \oplus \mathbb{Z}_n$ is cyclic iff $\gcd(r, n) = 1$.*

This lemma suggests that we can combine the prime powers for distinct primes for a different representation. Before we discuss the structure theorem, a helpful lemma:

**Lemma 3.** *Let $G$ be an abelian group, $m \in \mathbb{Z}$, $p$ a prime integer. Each of the following is a subgroup of $G$:*

1. $mG = \{mu \mid u \in G\}$
2. $G[m] = \{u \in G \mid mu = 0\}$
3. $G(p) = \{u \in G \mid \exists n \in \mathbb{N} \cup \{0\}, |u| = p^n\}$
4. $G_t = \{u \in G \mid |u| < \infty\}$

*Let $n, m$ be positive integers, $H$ and $G_i$ be abelian groups. There are isomorphisms*

5. *$\forall n, m, m < n, \mathbb{Z}_{p^n}[p] \cong \mathbb{Z}_p$ and $p^m\mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^{n-m}}$.*
6. *If $g : G \to \bigoplus_{i \in I} G_i$ is an isomorphism, the restrictions of $g$ to $mG$ and $G[m]$ respectively are isomorphisms $mG \cong \bigoplus_{i \in I} mG_i$, $G[m] \cong \bigoplus_{i \in I} G_i[m]$*
7. *If $f : G \to H$ is an isomorphism, the restrictions of $f$ to $G_t$ and $G(p)$ respectively are isomorphisms $G_t \cong H_t$ and $G(p) \cong H(p)$.*

Finally, we state the structure theorem.

**Theorem 1** (Structure Theorem for Finitely Generated Abelian Groups)**.** *Let $G$ be a finitely generated abelian group.*

1. *There is a unique nonnegative integer $s$ such that the number of infinite cyclic summands in any decomposition of $G$ as a direct sum of cyclic groups is precisely $s$.*

2. *Either $G$ is free abelian or there is a unique list of not necessarily distinct positive integers $m_1, \cdots, m_t$ such that $m_1 \mid m_2 \mid \cdots \mid m_t$ and*

$$G \cong \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_t} \oplus F$$

*with $F$ free abelian.*

3. *Either $G$ is free abelian or there is a list of positive integers $p_1^{s_1}, \cdots, p_k^{s_k}$ which is unique except for the order of its members such that $p_1, p_2, \cdots, p_k$ are primes that are not necessarily distinct and $s_1, s_2, \cdots, s_k$ are positive integers that are not necessarily distinct and*

$$G \cong \mathbb{Z}_{p_1^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}} \oplus F$$

*with $F$ free abelian.*

We omit the proof as it is quite lengthy, but it only uses elementary methods. If $G$ is a finitely generated abelian group, then the uniquely determined integers $m_1, m_2, \cdots, m_t$ are called the invariant factors of $G$. The uniquely determined prime powers are called the elementary divisors of $G$.

**Corollary 1.** *Two finitely generated abelian groups $G$ and $H$ are isomorphic iff $G/G_t$ and $H/H_t$ have the same rank and $G$ and $H$ have the same invariant factors (resp. elementary divisors).*

As an example, consider describing all finite abelian groups of order 1500 up to isomorphism. $1500 = 2^2 \cdot 3 \cdot 5^3$. The only possible families of elementary divisors are

$$\{2, 2, 3, 5^3\}, \{2, 2, 3, 5, 5^2\}, \{2, 2, 3, 5, 5, 5\}, \{2^2, 3, 5^3\}, \{2^2, 3, 5, 5^2\}, \{2^2, 3, 5, 5, 5\}.$$

In general, the number of families of elementary divisors depends on the integer partitions of the powers of the primes in the prime decomposition. Each of these six families determines an abelian group of order 1500. E.g. $\{2, 2, 3, 5^3\}$ determines $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{125}$. From the list of families of elementary divisors, we can derive an equivalent list of families of invariant factors, and vice versa. Suppose that an invariant factor decomposition $m_1, m_2, \cdots, m_t$ were known. Then the elementary divisors are the prime powers that arise from the prime factorizations of $m_1, m_2, \cdots, m_t$. Conversely, if the elementary divisors are known, one can arrange them in a matrix:

$$\begin{matrix} p_1^{n_{11}} & p_2^{n_{12}} & \cdots & p_r^{n_{1r}} \\ p_1^{n_{21}} & p_2^{n_{22}} & \cdots & p_r^{n_{2r}} \\ \vdots & \vdots & \ddots & \vdots \\ p_1^{n_{t1}} & p_2^{n_{t2}} & \cdots & p_r^{n_{tr}} \end{matrix}$$

where $p_1, p_2, \cdots, p_r$ are distinct primes, $\forall j, 0 \leq n_{1j} \leq n_{2j} \leq \cdots \leq n_{tj}$ with some $n_{ij} \neq 0$ and $n_{1j} \neq 0$ for some $j$. While this is a sufficient description, some observations that make this process easier to visualize: the last row contains the highest prime powers for each prime with no zero exponents. The first row contains nonzero exponents for the primes with the most amount of prime power terms in the family. Let $m_i = p_1^{n_{i1}} p_2^{n_{i2}} \cdots p_r^{n_{ir}}$. By construction, $m_1 \mid m_2 \mid \cdots \mid m_t$ and we have constructed the invariant factors.

As an example, consider $G = \mathbb{Z}_5 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{54}$. Then,

$$G \cong \mathbb{Z}_5 \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_{25} \oplus (\mathbb{Z}_4 \oplus \mathbb{Z}_9) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_{27})$$

The elementary divisors of $G$ are thus $2, 2^2, 3, 3^2, 3^3, 5, 5, 5^2$. This can be arranged as

$$\begin{matrix} 2^0 & 3 & 5 \\ 2 & 3^2 & 5 \\ 2^2 & 3^3 & 5^2 \end{matrix}$$

Thus the invariant factors of $G$ are $15, 90, 2700$ so that $G \cong \mathbb{Z}_{15} \oplus \mathbb{Z}_{90} \oplus \mathbb{Z}_{2700}$.

Here is a simple exercise from the section that I liked: prove that a finite abelian group that is not cyclic contains a subgroup which is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$ for some prime $p$.

Let $G$ be a finite abelian group that is not cyclic. Consider its invariant factor decomposition: $G \cong \bigoplus_{i=1}^t \mathbb{Z}_{m_i}$. If $t = 1$, then $G$ would be cyclic hence $t > 1$. Since $m_1 \mid m_2$ and $m_1 > 1$, $m_1$ and $m_2$ must share a prime factor $p$. Then $G[p] \cong \bigoplus_{i=1}^t \mathbb{Z}_p$. Take the subgroup corresponding to $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \bigoplus_{i=3}^t 0$.

In the next post, we discuss the Krull-Schmidt theorem which extends this notion of uniquely decomposing a group into a finite direct product of indecomposable subgroups to groups that satisfy both the ascending chain condition or descending chain condition on normal subgroups.