Products, coproducts, and free objects in groups and abelian groups

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Last post we discussed products, coproducts, and free objects generally in category theory. This post we focus our attention on **Grp** and **Ab**.

Definition 1. *For a family of groups $\{G_i \mid i \in I\}$, define a binary operation on the Cartesian product as follows. If $f, g \in \prod_{i \in I} G_i$, $fg : I \mapsto \bigcup_{i \in I} G_i$ is the function given by $i \mapsto f(i)g(i)$. $\prod_{i \in I} G_i$ is called the direct product of the family of groups.*

Definition 2. *The weak direct product of groups $\{G_i \mid i \in I\}$ denoted $\prod_{i \in I}^w G_i$ is the set of all $f \in \prod_{i \in I} G_i$ such that $f(i) = e_i$ for all but finitely many $i \in I$. If all the groups G_i are Abelian, $\prod_{i \in I}^w G_i$ is usually called the direct sum and denoted $\bigoplus_{i \in I} G_i$.*

It is trivial to prove that $\prod_{i \in I} G_i$ is a product in **Grp** and **Ab**. It is also easy to see that the direct sum of Abelian groups is a coproduct in **Grp**.

These definitions are external direct products or external direct sums, but sometimes a group has the direct product or direct sum structure within itself, and we may call it an internal weak direct product or internal direct sum in that case.

Theorem 1. *Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of G such that*

1. $G = \left\langle \bigcup_{i \in I} N_i \right\rangle$
2. $\forall k \in I, N_k \cap \left\langle \bigcup_{i \neq k} N_i \right\rangle = \langle e \rangle$

Then $G \cong \prod_{i \in I}^w N_i$.

Proof. If $(a_i)_{i \in I} \in \prod_{i \in I}^w N_i$, then $a_i = e_i$ except for finitely many $i \in I$. Let $I_0 = \{i \in I \mid a_i \neq e_i\}$. $\prod_{i \in I_0} a_i$ is a well-defined element of G, since for $a \in N_i, b \in N_j, i \neq j, ab = ba$. Consequently, $\phi : \prod_{i \in I}^w N_i \to G, a \mapsto \prod_{i \in I_0} a_i$ and $e \mapsto e$ is a homomorphism such that $\phi \iota_i(a_i) = a_i$ for $a_i \in N_i$. Since G is generated by the N_i's, every element $a \in G$ is a finite product of elements from various N_i. a can be expressed as $\prod_{i \in I_0} a_i$. Hence $\prod_{i \in I_0} \iota_i(a_i) \in \prod_{i \in I}^w N_i$ and $\phi\left(\prod_{i \in I_0} \iota_i(a_i)\right) = \prod_{i \in I_0} a_i = a$. Therefore ϕ is an epimorphism. Suppose that $\phi(a) = \prod_{i \in I_0} a_i = e \in G$. Assume for convenience that $I_0 = \{1, 2, \cdots, n\}$. Then $\prod_{i \in I_0} a_i = a_1 a_2 \cdots a_n = e$. Hence $a_1^{-1} = a_2 \cdots a_n \in N_1 \cap \langle \bigcup_{i=2}^n N_i \rangle = \langle e \rangle$ and therefore $a_1 = e$. Repetition of this argument shows $a_i = e$ for all $i \in I$. Hence ϕ is a monomorphism. □

In light of this theorem, we have the following definition:

Definition 3. *Let $\{N_i \mid i \in I\}$ be a family of normal subgroups of G such that $G = \left\langle \bigcup_{i\in I} N_i \right\rangle$ and $\forall k \in I, N_k \cap \left\langle \bigcup_{i\neq k} N_i \right\rangle = \langle e \rangle$. Then G is said to be the internal weak direct product of $\{N_i \mid i \in I\}$.*

We shall now construct a group F that is free on the set X. Let $X = \emptyset$, then F is the trivial group. If $X \neq \emptyset$, let X^{-1} be a set disjoint from X such that $|X| = |X^{-1}|$. Choose a bijection $X \to X^{-1}$ and denote the image of x by x^{-1}. Choose a singleton $\{1\}$ that is disjoint from $X \cup X^{-1}$. A word on X is a sequence $(a_1, a_2, \cdots)$ with $a_i \in X \cup X^{-1} \cup \{1\}$ such that for some $n \in \mathbb{N}$, $a_k = 1$ for all $k \geq n$. The constant sequence is called the empty word and is denoted 1. A word $(a_1, a_2, \cdots)$ on X is said to be reduced iff

1. $\forall x \in X$, x and x^{-1} are not adjacent
2. $a_k = 1$ implies $a_i = 1$ for all $i \geq k$.

Every nonempty reduced word is of the form $(x_1^{\lambda_1}, x_2^{\lambda_2}, \cdots, x_n^{\lambda_n}, 1, 1, \cdots)$ where $n \in \mathbb{N}$, $x_i \in X$, $\lambda_i \in \{-1, 1\}$. Hereafter, this word is denoted by $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$. Two reduced words $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m}$ and $y_1^{\delta_1} y_2^{\delta_2} \cdots y_n^{\delta_n}$ are equal iff both are 1 or $m = n$ and $x_i = y_i, \lambda_i = \delta_i$ for each $i \in \{1, 2, \cdots, n\}$. Consequently the map from X into the set $F(X)$ of all reduced words on X given by $x \mapsto x^1 = x$ is injective. Identify X with its image and consider it to be a subset of $F(X)$. Define a binary operation on the set $F = F(X)$ of reduced words on X by juxtaposition and cancellations of adjacent terms.

Theorem 2. *If X is a nonempty set, $F = F(X)$ is the set of all reduced words on X, then F is a group and $F = \langle X \rangle$.*

Proof. 1 is an identity element and $x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}$ has inverse $x_n^{-\delta_n} \cdots x_1^{-\delta_1}$. $\forall x \in X, \delta \in \{-1, 1\}$, let $|x^\delta| : F \to F$ be given by $1 \mapsto x^\delta$ and

$$x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n} \mapsto \begin{cases} x^\delta x_1^{\delta_1} \cdots x_n^{\delta_n} & x^\delta \neq x_1^{-\delta_1} \\ x_2^{\delta_2} \cdots x_n^{\delta_n} & x^\delta = x_1^{-\delta_1} \end{cases}$$

Let $A(F)$ be the group of permutations on F and F_0 the subgroup generated by $\{|x| \mid x \in X\}$. The map $\phi : F \to F_0$ given by $1 \mapsto \mathrm{id}_F$, $x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n} \mapsto |x_1^{\delta_1}| \cdots |x_n^{\delta_n}|$ is a surjection such that $\phi(w_1 w_2) = \phi(w_1)\phi(w_2)$. Since $1 \mapsto x_1^{\delta_1} x_2^{\delta_2} x_n^{\delta_n}$ under the map $|x_1^{\delta_1}| \cdots |x_n^{\delta_n}|$, ϕ is injective. The fact that F_0 is a group implies that associativity holds in F and that ϕ is an isomorphism of groups. □

Some facts about free groups: if $|X| \geq 2$, then the free group is nonabelian. Every element except 1 has infinite order. Every subgroup of a free group is itself a free group on some set.

Theorem 3. *Let F be the free group on a set X then F is a free object on the set X in* **Grp**.

Proof. Let G be a group and $f : X \to G$. Define $\bar{f} : F \to G$ to be $\bar{f}(1) = e$ and for $x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}$ a nonempty reduced word on X,

$$\bar{f}(x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}) = f(x_1)^{\delta_1} f(x_2)^{\delta_2} \cdots f(x_n)^{\delta_n}$$

$\bar{f}$ is a homomorphism such that $\bar{f} \circ \iota = f$. If $g : F \to G$ is any homomorphism such that $g \circ \iota = f$, then

$$g(x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}) = g(x_1)^{\delta_1} g(x_2)^{\delta_2} \cdots g(x_n)^{\delta_n} = \bar{f}(x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n})$$

Thus $\bar{f}$ is unique. □

Corollary 1. *Every group G is the homomorphic image of a free group.*

Proof. Let X be a set of generators of G and let F be the free group on the set X. The inclusion map $X \to G$ induces a homomorphism $\bar{f} : F \to G$ such that $x \mapsto x$. Since $G = \langle X \rangle$, $\bar{f}$ is an epimorphism. □

A consequence is that any group G is isomorphic to a quotient group F/N. F is the free group on X and N is the kernel of the epimorphism $F \to G$. F is determined to isomorphism by X and N is determined by any subset that generates it as a subgroup of F. If $w = x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n} \in F$ is a generator of N, then under the epimorphism $F \to G$, $w \mapsto x_1^{\delta_1} \cdots x_n^{\delta_n} = e$. The equation $x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n} = e$ in G is called a relation on the generators x_i. A given group G may be completely described by specifying a set X of generators of G and a suitable set R of relations on these generators. This description is not unique since there are many possible choices of both X and R. Conversely, suppose we are given a set X and a set Y of reduced words on the elements of X. Let F be a free group on X and N the normal subgroup of F generated by Y (intersection of all normal subgroups of F containing Y). Let $G = F/N$ and identify X with its image in F/N under the map $X \hookrightarrow F \twoheadrightarrow F/N$. Then G is a group generated by X and all the relations $w = e$ are satisfied.

Definition 4. *Let X be a set and Y a set of reduced words on X. A group G is said to be the group defined by the generators $x \in X$ and relations $w = e (w \in Y)$ provided $G \cong F/N$ where F is the free group on X and N is the normal subgroup of F generated by Y. One says that $(X \mid Y)$ is a presentation of G.*

Theorem 4 (Van Dyck's theorem)**.** *Let X be a set, Y a set of reduced words on X and G the group defined by generators $x \in X$ and relations $w = e, w \in Y$. If H is any group such that $H = \langle X \rangle$ and H satisfies all the relations $w = e$, then there is an epimorphism $G \to H$.*

Proof. If F is the free group on X then the inclusion map $X \to H$ induces an epimorphism $\phi : F \to H$. Since H satisfies the relations $w = e$, $Y \subseteq \ker \phi$. The normal subgroup N generated by Y in F is contained in $\ker \phi$. ϕ induces an epimorphism $F/N \to H/0$. Thus the composition is an epimorphism. □

Finally, we define the free product, which is the coproduct in **Grp**.

Given a family of groups $\{G_i \mid i \in I\}$, let $X = \coprod_{i\in I} G_i$. Let $\{1\}$ be a singleton disjoint from X. A word on X is any sequence $(a_1, a_2, \cdots)$ such that $a_i \in X \cup \{1\}$ and for some $n \in \mathbb{N}$, $a_i = 1$ for all $i \geq n$. A word $(a_1, a_2, \cdots)$ is reduced iff

1. No $a_i \in X$ is the identity element in its group G_i
2. $\forall i, j \geq 1, a_i$ and a_{i+1} are not in the same group G_j
3. $a_k = 1$ implies $a_i = 1$ for all $i \geq k$.

Let $\prod^*_{i\in I} G_i$ be the set of all reduced words on X. $\prod^*_{i\in I} G_i$ forms a group called the free product of $\{G_i \mid i \in I\}$. 1 is the identity element and the product of two words is juxtaposition with any necessary cancellations and contractions.

$\forall k \in I, \iota_k : G_k \to \prod^*_{i\in I} G_i$ given by $e \mapsto 1$, $a \mapsto a = (a, 1, 1, \cdots)$ is a monomorphism of groups.

Theorem 5. *Let $\{G_i \mid i \in I\}$ be a family of groups and $\prod^*_{i\in I} G_i$ their free product. Then $\prod^*_{i\in I} G_i$ is a coproduct in* **Grp**.

Proof. Let $\psi_i : G_i \to H$ be homomorphisms. If $a_1a_2 \cdots a_n$ is a reduced word in $\prod^*_{i\in I} G_i$ with $a_k \in G_{i_k}$, define

$$\psi(a_1a_2 \cdots a_n) = \psi_{i_1}(a_1)\psi_{i_2}(a_2) \cdots \psi_{i_n}(a_n) \in H$$

□

Finally, we discuss free abelian groups, the free objects in **Ab**.

Definition 5. *Let X be a nonempty subset of abelian group F. X is a basis of F iff*

1. *$F = \langle X\rangle$*
2. *For distinct $x_1, x_2, \cdots, x_k \in X$, $n_i \in \mathbb{Z}$, $n_1x_1 + n_2x_2 + \cdots n_kx_k = 0$ implies $n_i = 0$ for every i.*

Theorem 6. *The following conditions on an abelian group F are equivalent:*

1. *F has a nonempty basis*
2. *F is the internal direct sum of a family of infinite cyclic subgroups*
3. *F is isomorphic to a direct sum of $\mathbb{Z}$.*
4. *F is a free object on a nonempty set in* **Ab**.

The proof is straightforward but tedious.

Theorem 7. *Any two bases of a free abelian group F have the same cardinality.*

Proof. Suppose F has a basis X of finite cardinality n such that $F \cong \bigoplus_{i=1}^{n} \mathbb{Z}$. The restriction of the isomorphism to $2F$ is an isomorphism $2F \cong \bigoplus_{i=1}^{n} 2\mathbb{Z}$, whence $F/2F \cong \bigoplus_{i=1}^{n} \mathbb{Z}/2\mathbb{Z}$. Thus $|F/2F| = 2^n$. If Y is another basis of F and $|Y| = r \in \mathbb{N}$, then by a similar argument, $|F/2F| = 2^r$, whence $r = n$ and $|X| = |Y|$. Note that if Y was an infinite set, there would be a contradiction so Y had to be a finite set. Thus if one basis of F is infinite, then all bases are infinite. It suffices to show $|X| = |F|$ if X is any infinite basis of F. Clearly $|X| \leq |F|$. Let $S = \bigcup_{n\in\mathbb{N}} X^n$. For each $s = (x_1, x_2, \cdots, x_n) \in S$ let $G_s = \langle x_1, \cdots, x_n\rangle$. $G_s \cong \mathbb{Z}y_1 \oplus \cdots \mathbb{Z}y_t$ where $y_1, \cdots, y_t$ are the distinct elements in $\{x_1, x_2, \cdots, x_n\}$. Therefore $|G_s| = |\mathbb{Z}^t| = \aleph_0$. Since $F = \bigcup_{s\in S} G_s$, $|F| = \left|\bigcup_{s\in S} G_s\right| \leq |S|\aleph_0$. $|S| = |X|$, whence $|F| \leq |X|\aleph_0 = |X|$. Thus $|F| = |X|$ by Schroeder-Bernstein theorem. ☐

Theorem 8. *Every abelian group G is the homomorphic image of a free abelian group of rank $|X|$ where X is a set of generators of G.*

Proof. Let F be the free abelian group on the set X. Then $F = \bigoplus_{x\in X} \mathbb{Z}x$ and $\operatorname{rank} F = |X|$. The inclusion map $X \to G$ induces a homomorphism $\bar{f} : F \to G$ such that $1x \mapsto x$ whence $X \subseteq \operatorname{im} \bar{f}$. Since X generates G, we must have $\operatorname{im} \bar{f} = G$. ☐

In the next post, we discuss finitely generated abelian groups and their structures.