When semigroups are groups

- [Night Shift in Math](#)
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A semigroup is a nonempty set G with a binary operation on G that is associative. To be a group, G also needs an identity element and the existence of a two-sided inverse for each element.

Hungerford lists the following propositions that give conditions on when a semigroup is a group:

Proposition 1 *Let G be a semigroup. Then G is a group iff the following conditions hold:*

1. $\exists e \in G, \forall a \in G, ea = a$
2. $\forall a \in G, \exists a^{-1} \in G, a^{-1}a = e$

Proposition 2 *Let G be a semigroup. Then G is a group iff $\forall a, b \in G$ the equations $ax = b$ and $ya = b$ have solutions in G.*

Hungerford proves Proposition 1 and leaves as an exercise to show Proposition 2 holds from Proposition 1. This is not as simple as it may first appear. Having solutions of $ax = b, ya = b$ immediately gives elements that are left and right identities for particular elements of G, but we need all these identities to be equal to each other to satisfy the conditions of proposition 1. The second condition is achieved trivially.

Proof. Let $a \in G$. Find solutions $x, y \in G$ such that $ax = a, ya = a$. Then find solutions $z, w \in G$ such that $za = y$ and $aw = x$.

$$y = za = zax = yx = yaw = aw = x$$

Call this two-sided identity element of a, e_a. Note that

$$ae_a^2 = (ae_a)e_a = ae_a = a$$

So e_a is idempotent. Let $b \in G$ and find solutions $x, y \in G$ to $xe_b = e_a, e_ay = e_b$.

$$e_a = xe_b = xe_b^2 = e_ae_b = e_a^2y = e_ay = e_b$$

□

Here is a different characterization of a group:

Proposition 3 *A semigroup G is a group iff $\forall x \in G, \exists! y \in G, xyx = x$.*

Proof. For $x \in G$, denote by x' the unique element in G such that $xx'x = x$. Then,

$$x(x'xx')x = (xx'x)x'x = xx'x = x$$

Therefore, $x'xx' = x'$, so that $x'' = x$. Also note that xx' must be idempotent because

$$(xx')^2 = xx'xx' = xx'$$

For $i \in G$ an idempotent element, $ix = ix(ix)'ix = ix(ix)'i^2x$ so that $(ix)' = (ix)'i$. Therefore,

$$(ix)' = (ix)'(ix)''(ix)' = (ix)'ix(ix)' = (ix)'x(ix)'$$

Implying $ix = x$. Since x was arbitrary, this means that i is a left identity in G. With a symmetric argument, we can prove that i is also a right identity. Since i was arbitrary, all idempotents are equal to each other. In particular, xx' is an idempotent, hence x' is an inverse of x. $\square$