

Axler, Linear Algebra Done Right, Chapter 1.C Exercises

Dr. Heraklinos • 23 Apr 2024

Problem 4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Solution. Let $S \subseteq \mathbb{R}^{[0,1]}$ denote the set of continuous, real-valued functions f on $[0, 1]$ with the property that $\int_0^1 f \, dx = b$. Then $0 \in S$, so $\int_0^1 0 \, dx = 0 = b$. Conversely, suppose that $b = 0$. Given $f, g \in S$, we have

$$\int_0^1 (f + g) \, dx = \int_0^1 f \, dx + \int_0^1 g \, dx = 0 + 0 = 0 = b,$$

so $f + g \in S$. We have $\int_0^1 0 \, dx = 0 = b$, so $0 \in S$. Finally, if $f \in S$ and $\alpha \in \mathbb{R}$, one has

$$\int_0^1 (\alpha f) \, dx = \alpha \int_0^1 f \, dx = \alpha \cdot 0 = 0 = b,$$

so $\alpha f \in S$. So S is a subspace of $\mathbb{R}^{[0,1]}$, as required.

Problem 7. Prove or give a counterexample: If U is a nonempty subset of \mathbb{R}^2 such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of \mathbb{R}^2 .

Proof. Let $U = \mathbb{Z}^2$, which is a nonempty subset of \mathbb{R}^2 . Then, as \mathbb{Z} is closed under additive inverse, if $(a, b) \in U$, then $(-a, -b) \in U$; but $(a, b) + (-a, -b) = (a + (-a), b + (-b)) = (0, 0)$, so U is closed under taking inverses. However, notice that U is not closed under scalar multiplication. Indeed, we have $(1, 1) \in \mathbb{Z}^2$ and $\frac{1}{2} \in \mathbb{R}$, but $\frac{1}{2}(1, 1) = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^2 - U$. So U is not a subspace of \mathbb{R}^2 .