Axler, Linear Algebra Done Right, Chapter 1.C Exercises

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Problem 4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Solution. Let $S \subseteq \mathbb{R}^{[0,1]}$ denote the set of continuous, real-valued functions f on $[0,1]$ with the property that $\int_0^1 f \, \mathrm{d}x = b$. Then $0 \in S$, so $\int_0^1 0 \, \mathrm{d}x = 0 = b$. Conversely, suppose that $b = 0$. Given$f, g \in S$, we have

$$\int_0^1 (f+g) \, \mathrm{d}x = \int_0^1 f \, \mathrm{d}x + \int_0^1 g \, \mathrm{d}x = 0 + 0 = 0 = b,$$

so $f + g \in S$. We have $\int_0^1 0 \, \mathrm{d}x = 0 = b$, so $0 \in S$. Finally, if $f \in S$ and $\alpha \in \mathbb{R}$, one has

$$\int_0^1 (\alpha f) \, \mathrm{d}x = \alpha \int_0^1 f \, \mathrm{d}x = \alpha \cdot 0 = 0 = b,$$

so $\alpha f \in S$. So S is a subspace of $\mathbb{R}^{[0,1]}$, as required.

Problem 7. Prove or give a counterexample: If U is a nonempty subset of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbb{R}^2$.

Proof. Let $U = \mathbb{Z}^2$, which is a nonempty subset of $\mathbb{R}^2$. Then, as $\mathbb{Z}$ is closed under additive inverse, if $(a,b) \in U$, then $(-a,-b) \in U$; but $(a,b) + (-a,-b) = (a+(-a), b+(-b)) = (0,0)$, so U is closed under taking inverses. However, notice that U is not closed under scalar multiplication. Indeed, we have $(1,1) \in \mathbb{Z}^2$ and $\frac{1}{2} \in \mathbb{R}$, but $\frac{1}{2}(1,1) = \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R} - U$. So U is not a subspace of $\mathbb{R}^2$.