Axler, Linear Algebra Done Right, Chapter 1.B Exercises

written by Dr. Heraklinos on Functor Network
original link: https://functor.network/user/842/entry/326

Problem 1. Prove that $-(-v) = v$ for every $v \in V$.

Solution. By definition we have $-v + (-(-v)) = 0$. But $-v + v = 0$. So by uniqueness of the additive inverse (1.27), we conclude that $-(-v) = -v$.

Problem 2. Suppose $a \in \mathbb{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution. Suppose that $a \neq 0$. Then a admits a multiplicative inverse $a^{-1} \in \mathbb{F}$, and it follows that

$$v = 1v = \left(a^{-1}a\right) v = a^{-1}(av) = a^{-1}0 = 0$$

by (1.31).

Problem 3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Proof. Let $x = \frac{1}{3}(w - v)$. This vector exists because v admits an additive inverse in V and V is closed under addition and scalar multiplication. It then follows that

$$v + 3x = v + 3\left(\frac{1}{3}(w - v)\right) = v + (w - v) = w,$$

which establishes the existence of such an x. Now, suppose that $v + 3x = w = v + 3x'$ for $x, x' \in V$. Adding $-v$ on the left and cancelling yields $3x = 3x'$. Scaling these vector by $\frac{1}{3}$ yields $x = x'$, so this vector is unique.

Problem 4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Solution. The empty set contains no elements, and hence it fails to admit an additive identity element. So a vector space V, as a set, must be nonempty.

Problem 5. Show that in the definition of a vector space (1.20) , the additive inverse condition can be replaced with the condition that $0v = 0$ for all $v \in V$. Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

Solution. Given the vector-space axioms, one has $0v = 0$ for every $v \in V$ by (1.30). Conversely, suppose that the additive-inverse axiom were replaced with the condition that $0v = 0$ for every $v \in V$. Fix $v \in V$. Then $(-1)v \in V$. But $(-1)v = -v$ by (1.32), so one has

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0,$$

meaning that v admits an additive inverse, as required.