Stillwell, Elements of Number Theory, Chapter 1 Exercises

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Problem 1.1.1 Check that the quadratic function $n^2 + n + 41$ is prime for all small values of n (say, for n up to 30).

Solution. Using the matlab package in R, we check whether the quantity n^2+n+41 is prime for each positive natural number less than or equal to 41. We find that, for each n less than or equal to 40, this quantity lacks a proper divisor other than 1 and is hence prime.

n	n^2+n+41	Prime or Composite
1	43	Prime
2	47	Prime
3	53	Prime
4	61	Prime
5	71	Prime
6	83	Prime
7	97	Prime
8	113	Prime
9	131	Prime
10	151	Prime
11	173	Prime
12	197	Prime
13	223	Prime
14	251	Prime
15	281	Prime
16	313	Prime
17	347	Prime
18	383	Prime
19	421	Prime
20	461	Prime
21	503	Prime
22	547	Prime
23	593	Prime
24	641	Prime
25	691	Prime
26	743	Prime
27	797	Prime
28	853	Prime
29	911	Prime
30	971	Prime
31	$1,033$	Prime
32	$1,097$	Prime
33	$1,163$	Prime
34	$1,231$	Prime
35	$1,301$	Prime
36	$1,373$	Prime
37	$1,447$	Prime
38	$1,523$	Prime
39	$1,601$	Prime
40	$1,681$	Composite
41	$1,763$	Composite

Problem 1.2. Show nevertheless that n^2+n+41 is not prime for certain values of n.

Solution. Let $n = 41$. Then

$$41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \cdot 43,$$

meaning that $n^2 + n + 41$ has two non-trivial proper divisor and is therefore composite.

Problem 1.3. What is the smallest such value?

Solution. Using the matlab package in R and the table we generated in Problem 1.1.1, we find that $n^2 + n + 41$ is prime provided that $n \leq 39$. But when $n = 40$, we have $40^2 + 40 + 41 = 1681$, which is composite. In particular, it has a proper divisor of 41 as $1681 = 41^2$. So $n = 40$ is the smallest such value of n for which $n^2 + n + 41$ is not prime.