Munkres, Topology, Section 13 Exercises

written by Dr. Heraklinos on Functor Network
original link: https://functor.network/user/842/entry/321

Problem 1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Solution. For each $x \in A$, choose an open subset $U_x \subseteq A$ such that $x \in U_x$. I claim that $\bigcup_{x \in A} U_x = A$. Indeed, we have $U_x \subseteq A$ for each $x \in A$ by definition, so $\bigcup_{x \in A} U_x \subseteq A$. Furthermore, for each $x \in A$, one has $x \in U_x \subseteq \bigcup_{x \in A} U_x$, which gives the opposite inclusion. As each U_x is open and arbitrary unions of open sets are open, the union $\bigcup_{x \in X} U_x$ is open in X and hence A is open in X.