

# Rudin, Principles of Mathematical Analysis, Chapter 3 Exercises

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**Problem 3.1.** Prove that the convergence of  $\{s_n\}$  implies convergence of  $\{|s_n|\}$ .  
Is the converse true?

*Solution.* Define  $s := \lim_{n \rightarrow \infty} s_n$ . I claim that  $(|s_n|)_n$  converges to  $|s|$ . Fix  $\epsilon > 0$ .  
As  $(s_n)_n \rightarrow s$ , there exists  $N > 0$  such that  $|s_n - s| < \epsilon$  for any  $n \geq N$ . Then  
for any  $n \geq N$ , one has

$$||s_n| - |s|| \leq |s_n - s| < \epsilon$$

by Problem 1.13. So  $(|s_n|)_n$  converges to  $|s|$ .

The converse is not however true. Let  $(s_n)_n := (-1)^n$ . Then  $(|s_n|)_n = 1$  and  
hence  $\lim_{n \rightarrow \infty} |s_n| = 1$ . But  $(s_n)$  fails to converge. Indeed, suppose for the sake of  
contradiction that  $\lim_{n \rightarrow \infty} s_n = \ell$  for some  $\ell \in \mathbb{C}$ . Then there exists  $N > 0$  such  
that for every  $n \geq N$ , one has  $|s_n - \ell| = |(-1)^n - \ell| < 1$ . Then  $2N > N$  is  
even, so  $|(-1)^{2N} - \ell| = |1 - \ell| < 1$  and hence  $\ell \in (0, 2)$ . Furthermore, as  
 $2N + 1 > N$  is odd, we have  $|(-1)^{2N+1} - \ell| = |-1 - \ell| < 1$  and hence  
 $\ell \in (-2, 0)$ . But  $(0, 2) \cap (-2, 0) = \emptyset$ , so we have reached a contradiction. So  
the sequence  $(s_n)_n$  fails to converge in spite of the fact that  $(|s_n|)_n$  converges.

**Problem 3.2.** Calculate  $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$ .

*Solution.* For any  $n \in \mathbb{N}$ , we have that

$$\sqrt{n^2 + n} - n = (\sqrt{n^2 + n} - n) \cdot \frac{\sqrt{n^2 + n} + n}{\sqrt{n^2 + n} + n} = \frac{(n^2 + n) - n^2}{\sqrt{n^2 + n} + n} = \frac{n}{\sqrt{n^2 + n} + n}.$$

Dividing the numerator and denominator by  $n = \sqrt{n^2}$ , we find that

$$\frac{n}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + \frac{1}{n^2}} + 1}.$$

*Lemma.* Let  $(s_n)_n$  be a sequence of non-negative real numbers which converges  
to  $s > 0$ . Then  $(\sqrt{s_n})_n$  converges to  $\sqrt{s}$ .

*Proof.* Fix  $\epsilon > 0$ . As  $(s_n)_n \rightarrow s$ , there exist  $N > 0$  such that  $|s_n - s| < \sqrt{s}\epsilon$  for  
every  $n \geq N$ . Then, for any  $n \geq N$ , one has

$$|\sqrt{s_n} - \sqrt{s}| = \left| \sqrt{s_n} - \sqrt{s} \cdot \frac{\sqrt{s_n} + \sqrt{s}}{\sqrt{s_n} + \sqrt{s}} \right| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s}\epsilon}{\sqrt{s}} = \epsilon,$$

so  $(\sqrt{s_n})_n \rightarrow \sqrt{s}$ . ■

Now, as  $(\frac{1}{n})_n \rightarrow 0$ , Theorem 3.3(a) and the lemma imply that

$\lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n^2}} = 1$ . So by Theorem 3.3(a), (c), and (d), we find that

$\frac{1}{\sqrt{1 + \frac{1}{n^2} + 1}} = \frac{1}{2}$ . So  $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n) = \frac{1}{2}$ , as required.