Rudin, Principles of Mathematical Analysis, Chapter 3 Exercises

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Problem 3.1. Prove that the convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

Solution. Define $s \coloneqq \lim_{n\to\infty} s_n$. I claim that $(|s_n|)_n$ converges to $|s|$. Fix $\epsilon > 0$. As $(s_n)_n \to s$, there exists $N > 0$ such that $|s_n - s| < \epsilon$ for any $n \geq N$. Then for any $n \geq N$, one has

$$||s_n| - |s|| \leq |s_n - s| < \epsilon$$

by Problem 1.13. So $(|s_n|)_n$ converges to $|s|$.

The converse is not however true. Let $(s_n)_n \coloneqq (-1)^n$. Then $(|s_n|)_n = 1$ and hence $\lim_{n\to\infty} |s_n| = 1$. But (s_n) fails to converge. Indeed, suppose for the sake of contradiction that $\lim_{n\to\infty} s_n = \ell$ for some $\ell \in \mathbb{C}$. Then there exists $N > 0$ such that for every $n \geq N$, one has $|s_n - \ell| = |(-1)^n - \ell| < 1$. Then $2N > N$ is even, so $|(-1)^{2N} - \ell| = |1 - \ell| < 1$ and hence $\ell \in (0, 2)$. Furthermore, as $2N + 1 > N$ is odd, we have $|(-1)^{2N+1} - \ell| = |-1 - \ell| < 1$ and hence $\ell \in (-2, 0)$. But $(0, 2) \cap (-2, 0) = \emptyset$, so we have reached a contradiction. So the sequence $(s_n)_n$ fails to converge in spite of the fact that $(|s_n|)_n$ converges.

Problem 3.2. Calculate $\lim_{n\to\infty} (\sqrt{n^2 + n} - n)$.

Solution. For any $n \in \mathbb{N}$, we have that

$$\sqrt{n^2+n} - n = (\sqrt{n^2+n} - n) \cdot \frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n} = \frac{(n^2+n) - n^2}{\sqrt{n^2+n} + n} = \frac{n}{\sqrt{n^2+n} + n}.$$

Dividing the numerator and denominator by $n = \sqrt{n^2}$, we find that

$$\frac{n}{\sqrt{n^2+n} + n} = \frac{1}{\sqrt{1 + \frac{1}{n^2}} + 1}.$$

Lemma. Let $(s_n)_n$ be a sequence of non-negative real numbers which converges to $s > 0$. Then $(\sqrt{s_n})_n$ converges to $\sqrt{s}$.

Proof. Fix $\epsilon > 0$. As $(s_n)_n \to s$, there exist $N > 0$ such that $|s_n - s| < \sqrt{s}\epsilon$ for every $n \geq N$. Then, for any $n \geq N$, one has

$$|\sqrt{s_n} - \sqrt{s}| = \left|\sqrt{s_n} - \sqrt{s} \cdot \frac{\sqrt{s_n} + \sqrt{s}}{\sqrt{s_n} + \sqrt{s}}\right| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s}\epsilon}{\sqrt{s}} = \epsilon,$$

so $(\sqrt{s_n})_n \to \sqrt{s}$. ■

Now, as $\left(\frac{1}{n}\right)_n \to 0$, Theorem 3.3(a) and the lemma imply that $\lim_{n\to\infty} \sqrt{1 + \frac{1}{n^2}} = 1$. So by Theorem 3.3(a), (c), and (d), we find that $\frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{2}$. So $\lim_{n\to\infty} (\sqrt{n^2+n} - n) = \frac{1}{2}$, as required.