Rudin, Principles of Mathematical Analysis, Chapter 1 Exercises

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Problem 1.1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution. For the sake of contradiction, suppose that $r + x \in \mathbb{Q}$. Then, as $\mathbb{Q}$ is a field, we have $-r \in \mathbb{Q}$ by (A5) and hence $-r+(r+x) \in \mathbb{Q}$ by (M1). But then $-r + (r + x) = x \in \mathbb{Q}$, contradicting the fact that x is irrational. Analogously, suppose that $rx \in \mathbb{Q}$. As $r \neq 0$, there exists a multiplicative inverse $\frac{1}{r} \in \mathbb{Q}$. So $\frac{1}{r} \cdot (rx) \in \mathbb{Q}$ by (M1), but $\frac{1}{r} \cdot (rx) = x$, so $x \in \mathbb{Q}$. This is, once more, a contradiction, so we conclude that both $r + x$ and rx are irrational, as required.

Problem 1.2. Prove that there is no rational number whose square is 12.

Solution. Suppose, for the sake of contradiction, that there exists $x \in \mathbb{Q}$ satisfying $x^2 = 12$. Then write $x = \frac{m}{n}$ for $m, n \in \mathbb{Z}$ and $n \neq 0$. Dividing the numerator and denominator by $\gcd(m, n)$ if necessary, we can assume that m and n are relatively prime. Then

$$x^2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2} = 12$$

and hence $m^2 = 12n^2 = 3(4n^2)$. So 3 divides m^2. As 3 is prime, Euclid's lemma then implies that 3 divides m. So $m = 3k$ for some $k \in \mathbb{Z}$. So we find that

$$(3k)^2 = 9k^2 = 12n^2,$$

and hence $3k^2 = 4n^2$. So 3 divides $4n^2$. But then, as 3 does not divide 4, we must have that 3 divides n^2 and hence 3 divides n by Euclid's lemma. But then m and n share a common factor of 3, contradicting the fact that they were chosen to be relatively prime.

Problem 1.3. Prove Proposition 1.15.

Solution. (a) Let $x, y, z \in \mathbb{F}$ with $x \neq 0$, and suppose that $xy = xz$. By

(M5), there exists a multiplicative inverse $\frac{1}{x} \in \mathbb{F}$ of x. We then find that

$$\begin{aligned}
y &= 1y && \text{(M4)} \\
&= \left(x \cdot \frac{1}{x}\right) y && \text{(M5)} \\
&= \left(\frac{1}{x} \cdot x\right) y && \text{(M2)} \\
&= \frac{1}{x}(xy) && \text{(M3)} \\
&= \frac{1}{x}(xz) && \text{assumption that } xy = xz \\
&= \left(\frac{1}{x} \cdot x\right) z && \text{(M3)} \\
&= \left(x \cdot \frac{1}{x}\right) z && \text{(M2)} \\
&= 1z && \text{(M5)} \\
&= z. && \text{(M4)}
\end{aligned}$$

(b) Applying part (a) with $z = 1$ gives the result.

(c) Applying part (a) with $z = \frac{1}{x}$ gives the result.

(d) We apply part (c), replacing x with $\frac{1}{x}$ (which is likewise non-zero) and y with x. We then find that $x = 1/(1/x)$.

Problem 1.4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution. As E is nonempty, we may choose $p \in E$. Then $p \geq \alpha$ since α is a lower bound of E, and $p \leq \beta$ since β is an upper bound of E. This reduces to four cases. If $p = \alpha$ and $p = \beta$, then $\alpha = \beta$. If $p > \alpha$ and $p < \beta$, then $\alpha < \beta$ by the ordered-field axioms. If $p = \alpha$ and $p < \beta$, then substituting yields $\alpha < \beta$. Finally, if $p = \beta$ and $p > \alpha$, then substituting yields $\beta > \alpha$. So we conclude that $\alpha \leq \beta$, as required.

Problem 1.8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint* : -1 is a square.

Solution. Suppose, seeking a contradiction, that there exists an order $<$ on $\mathbb{C}$ that turns it into an ordered field. Then $i \neq 0$, so by Proposition 1.18(d), we have $i^2 > 0$. As $i^2 = -1$ by Proposition 2.28, we have $-1 > 0$. The ordered-field axioms then imply that $1 + (-1) > 1 + 0$ and hence $0 > 1$. This contradicts the trichotomy of order, as 1 is a square and hence strictly greater than 0 by Proposition 1.18(d).

Problem 1.12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|.$$

Solution. We proceed by induction on n. When $n = 1$, the statement reduces to $|z_1| \leq |z_1|$. The $n = 2$ case is given by Theorem 1.33(e). Suppose inductively that we have

$$\left|\sum_{i=1}^{k} z_i\right| \leq \sum_{i=1}^{k} |z_i|$$

for a fixed $k \geq 1$. It then follows that

$$\begin{aligned}
\left|\sum_{i=1}^{k+1} z_i\right| &= \left|\sum_{i=1}^{k} z_i + z_{k+1}\right| \\
&\leq \left|\sum_{i=1}^{k} z_i\right| + |z_{k+1}| \\
&\leq \sum_{i=1}^{k} |z_i| + |z_{k+1}| \\
&= \sum_{i=1}^{k+1} |z_i|,
\end{aligned}$$

which closes the induction.

Problem 1.13. If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

Solution. Let $x, y \in \mathbb{C}$. By the triangle inequality (Theorem 1.33(e)), we find that

$$|x| = |(x - y) + y| \leq |x - y| + |y|,$$

hence

$$|x| - |y| \leq |x - y|.$$

Analogously, one has

$$|y| = |(y - x) + x| \leq |y - x| + |x| = |x - y| + |x|,$$

hence

$$|y| - |x| = -(|x| - |y|) \leq |x - y|,$$

from which it follows that

$$-|x - y| \leq |x| - |y|.$$

We therefore find that

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

so it follows that

$$||x| - |y|| \leq |x - y|,$$

as required.

Problem 1.17. Prove that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{x}|^2$$

if $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement of parallelograms.

Solution. Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, we obtain

$$\begin{aligned}
|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \\
&= \mathbf{x}\cdot(\mathbf{x}+\mathbf{y}) + \mathbf{y}\cdot(\mathbf{x}+\mathbf{y}) + \mathbf{x}\cdot(\mathbf{x}-\mathbf{y}) + (-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \\
&= \mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} + \mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot(-\mathbf{y}) + (-\mathbf{y})\cdot\mathbf{x} + (-\mathbf{y})\cdot(-\mathbf{y}) \\
&= (\mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{x}) + (\mathbf{y}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{x}\cdot(-\mathbf{y}) + (-\mathbf{y})\cdot\mathbf{x}) \\
&= 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} - 2\mathbf{x}\cdot\mathbf{y} \\
&= 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2.
\end{aligned}$$

Geometrically, this is a statement about the k-parallelogram in $\mathbb{R}^k$ spanned by $\mathbf{x}$ and $\mathbf{y}$. This parallelogram has diagonals $|\mathbf{x}+\mathbf{y}|$ and $|\mathbf{x}-\mathbf{y}|$, two sides of length $|\mathbf{x}|$, and two sides of length $|\mathbf{y}|$. This statement asserts that the sum of the squared lengths of the diagonals is equal to the sum of the squared lengths of the four sides of the parallelogram.

Problem 1.18. If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x}\cdot\mathbf{y} = 0$. Is this also true if $k = 1$?

Solution. If $\mathbf{x} = \mathbf{0} \in \mathbb{R}^k$, then for any non-zero $\mathbf{y} \in \mathbb{R}^k$, one has $\mathbf{x}\cdot\mathbf{y} = 0$. Suppose that $\mathbf{x} = (x_1, \ldots, x_k) \neq \mathbf{0}$. Then there exists i such that $x_i \neq 0$. As dot products are invariant under permutation of indices, we can assume that $x_1 \neq 0$. Now define $\mathbf{y} = (-x_2, x_1, 0, \ldots, 0)$. Then $\mathbf{y} \neq \mathbf{0}$, but

$$\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^{k} x_i y_i = -x_1 x_2 + x_1 x_1 + 0 = 0.$$

The result is no longer true if $k = 1$. Indeed, in $\mathbb{R}^1$, the dot product coincides exactly with multiplication of real scalars. But $\mathbb{R}$ is a field and hence lacks zero divisors, so $xy = 0$ if and only if $x = 0$ or $y = 0$. If $x = 0$, then again any y will suffice. If $x \neq 0$, then $xy = 0$ only if $y = 0$.