

# Rudin, Principles of Mathematical Analysis, Chapter 1 Exercises

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**Problem 1.1.** If  $r$  is rational ( $r \neq 0$ ) and  $x$  is irrational, prove that  $r + x$  and  $rx$  are irrational.

*Solution.* For the sake of contradiction, suppose that  $r + x \in \mathbb{Q}$ . Then, as  $\mathbb{Q}$  is a field, we have  $-r \in \mathbb{Q}$  by (A5) and hence  $-r + (r + x) \in \mathbb{Q}$  by (M1). But then  $-r + (r + x) = x \in \mathbb{Q}$ , contradicting the fact that  $x$  is irrational. Analogously, suppose that  $rx \in \mathbb{Q}$ . As  $r \neq 0$ , there exists a multiplicative inverse  $\frac{1}{r} \in \mathbb{Q}$ . So  $\frac{1}{r} \cdot (rx) \in \mathbb{Q}$  by (M1), but  $\frac{1}{r} \cdot (rx) = x$ , so  $x \in \mathbb{Q}$ . This is, once more, a contradiction, so we conclude that both  $r + x$  and  $rx$  are irrational, as required.

**Problem 1.2.** Prove that there is no rational number whose square is 12.

*Solution.* Suppose, for the sake of contradiction, that there exists  $x \in \mathbb{Q}$  satisfying  $x^2 = 12$ . Then write  $x = \frac{m}{n}$  for  $m, n \in \mathbb{Z}$  and  $n \neq 0$ . Dividing the numerator and denominator by  $\gcd(m, n)$  if necessary, we can assume that  $m$  and  $n$  are relatively prime. Then

$$x^2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2} = 12$$

and hence  $m^2 = 12n^2 = 3(4n^2)$ . So 3 divides  $m^2$ . As 3 is prime, Euclid's lemma then implies that 3 divides  $m$ . So  $m = 3k$  for some  $k \in \mathbb{Z}$ . So we find that

$$(3k)^2 = 9k^2 = 12n^2,$$

and hence  $3k^2 = 4n^2$ . So 3 divides  $4n^2$ . But then, as 3 does not divide 4, we must have that 3 divides  $n^2$  and hence 3 divides  $n$  by Euclid's lemma. But then  $m$  and  $n$  share a common factor of 3, contradicting the fact that they were chosen to be relatively prime.

**Problem 1.3.** Prove Proposition 1.15.

*Solution.* (a) Let  $x, y, z \in \mathbb{F}$  with  $x \neq 0$ , and suppose that  $xy = xz$ . By (M5), there exists a multiplicative inverse  $\frac{1}{x} \in \mathbb{F}$  of  $x$ . We then find that

$$\begin{aligned}
y &= 1y && \text{(M4)} \\
&= \left(x \cdot \frac{1}{x}\right) y && \text{(M5)} \\
&= \left(\frac{1}{x} \cdot x\right) y && \text{(M2)} \\
&= \frac{1}{x}(xy) && \text{(M3)} \\
&= \frac{1}{x}(xz) && \text{assumption that } xy = xz \\
&= \left(\frac{1}{x} \cdot x\right) z && \text{(M3)} \\
&= \left(x \cdot \frac{1}{x}\right) z && \text{(M2)} \\
&= 1z && \text{(M5)} \\
&= z. && \text{(M4)}
\end{aligned}$$

(b) Applying part (a) with  $z = 1$  gives the result.

(c) Applying part (a) with  $z = \frac{1}{x}$  gives the result.

(d) We apply part (c), replacing  $x$  with  $\frac{1}{x}$  (which is likewise non-zero) and  $y$  with  $x$ . We then find that  $x = 1/(1/x)$ .

**Problem 1.4.** Let  $E$  be a nonempty subset of an ordered set; suppose  $\alpha$  is a lower bound of  $E$  and  $\beta$  is an upper bound of  $E$ . Prove that  $\alpha \leq \beta$ .

*Solution.* As  $E$  is nonempty, we may choose  $p \in E$ . Then  $p \geq \alpha$  since  $\alpha$  is a lower bound of  $E$ , and  $p \leq \beta$  since  $\beta$  is an upper bound of  $E$ . This reduces to four cases. If  $p = \alpha$  and  $p = \beta$ , then  $\alpha = \beta$ . If  $p > \alpha$  and  $p < \beta$ , then  $\alpha < \beta$  by the ordered-field axioms. If  $p = \alpha$  and  $p < \beta$ , then substituting yields  $\alpha < \beta$ . Finally, if  $p = \beta$  and  $p > \alpha$ , then substituting yields  $\beta > \alpha$ . So we conclude that  $\alpha \leq \beta$ , as required.

**Problem 1.8.** Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint* :  $-1$  is a square.

*Solution.* Suppose, seeking a contradiction, that there exists an order  $<$  on  $\mathbb{C}$  that turns it into an ordered field. Then  $i \neq 0$ , so by Proposition 1.18(d), we have  $i^2 > 0$ . As  $i^2 = -1$  by Proposition 2.28, we have  $-1 > 0$ . The ordered-field axioms then imply that  $1 + (-1) > 1 + 0$  and hence  $0 > 1$ . This contradicts the trichotomy of order, as  $1$  is a square and hence strictly greater than  $0$  by Proposition 1.18(d).

**Problem 1.12.** If  $z_1, \dots, z_n$  are complex, prove that

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|.$$

*Solution.* We proceed by induction on  $n$ . When  $n = 1$ , the statement reduces to  $|z_1| \leq |z_1|$ . The  $n = 2$  case is given by Theorem 1.33(e). Suppose inductively that we have

$$\left| \sum_{i=1}^k z_i \right| \leq \sum_{i=1}^k |z_i|$$

for a fixed  $k \geq 1$ . It then follows that

$$\begin{aligned} \left| \sum_{i=1}^{k+1} z_i \right| &= \left| \sum_{i=1}^k z_i + z_{k+1} \right| \\ &\leq \left| \sum_{i=1}^k z_i \right| + |z_{k+1}| \\ &\leq \sum_{i=1}^k |z_i| + |z_{k+1}| \\ &= \sum_{i=1}^{k+1} |z_i|, \end{aligned}$$

which closes the induction.

**Problem 1.13.** If  $x, y$  are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

*Solution.* Let  $x, y \in \mathbb{C}$ . By the triangle inequality (Theorem 1.33(e)), we find that

$$|x| = |(x - y) + y| \leq |x - y| + |y|,$$

hence

$$|x| - |y| \leq |x - y|.$$

Analogously, one has

$$|y| = |(y - x) + x| \leq |y - x| + |x| = |x - y| + |x|,$$

hence

$$|y| - |x| = -(|x| - |y|) \leq |x - y|,$$

from which it follows that

$$-|x - y| \leq |x| - |y|.$$

We therefore find that

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

so it follows that

$$||x| - |y|| \leq |x - y|,$$

as required.

**Problem 1.17.** Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if  $\mathbf{x} \in \mathbb{R}^k$  and  $\mathbf{y} \in \mathbb{R}^k$ . Interpret this geometrically, as a statement of parallelograms.

*Solution.* Given  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ , we obtain

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) + (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\ &= \mathbf{x} \cdot (\mathbf{x} + \mathbf{y}) + \mathbf{y} \cdot (\mathbf{x} + \mathbf{y}) + \mathbf{x} \cdot (\mathbf{x} - \mathbf{y}) + (-\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot (-\mathbf{y}) + (-\mathbf{y}) \cdot \mathbf{x} + (-\mathbf{y}) \cdot (-\mathbf{y}) \\ &= (\mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{x}) + (\mathbf{y} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{x} \cdot (-\mathbf{y}) + (-\mathbf{y}) \cdot \mathbf{x}) \\ &= 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2 + 2\mathbf{x} \cdot \mathbf{y} - 2\mathbf{x} \cdot \mathbf{y} \\ &= 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2. \end{aligned}$$

Geometrically, this is a statement about the  $k$ -parallelogram in  $\mathbb{R}^k$  spanned by  $\mathbf{x}$  and  $\mathbf{y}$ . This parallelogram has diagonals  $|\mathbf{x} + \mathbf{y}|$  and  $|\mathbf{x} - \mathbf{y}|$ , two sides of length  $|\mathbf{x}|$ , and two sides of length  $|\mathbf{y}|$ . This statement asserts that the sum of the squared lengths of the diagonals is equal to the sum of the squared lengths of the four sides of the parallelogram.

**Problem 1.18.** If  $k \geq 2$  and  $\mathbf{x} \in \mathbb{R}^k$ , prove that there exists  $\mathbf{y} \in \mathbb{R}^k$  such that  $\mathbf{y} \neq \mathbf{0}$  but  $\mathbf{x} \cdot \mathbf{y} = 0$ . Is this also true if  $k = 1$ ?

*Solution.* If  $\mathbf{x} = \mathbf{0} \in \mathbb{R}^k$ , then for any non-zero  $\mathbf{y} \in \mathbb{R}^k$ , one has  $\mathbf{x} \cdot \mathbf{y} = 0$ .

Suppose that  $\mathbf{x} = (x_1, \dots, x_k) \neq \mathbf{0}$ . Then there exists  $i$  such that  $x_i \neq 0$ . As dot products are invariant under permutation of indices, we can assume that  $x_1 \neq 0$ . Now define  $\mathbf{y} = (-x_2, x_1, 0, \dots, 0)$ . Then  $\mathbf{y} \neq \mathbf{0}$ , but

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^k x_i y_i = -x_1 x_2 + x_1 x_1 + 0 = 0.$$

The result is no longer true if  $k = 1$ . Indeed, in  $\mathbb{R}^1$ , the dot product coincides exactly with multiplication of real scalars. But  $\mathbb{R}$  is a field and hence lacks zero divisors, so  $xy = 0$  if and only if  $x = 0$  or  $y = 0$ . If  $x = 0$ , then again any  $y$  will suffice. If  $x \neq 0$ , then  $xy = 0$  only if  $y = 0$ .