# Tensor Products - How two polynomial rings in one variable give one polynomial ring in two variables! (huh?)

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Let $R$ be a ring. The set of all polynomials in the variable $x$ with coefficients in the ring $R$ is denoted by $R[x]$. An example of an element in the set $R[x]$ would be $x^3 + 5$.

Some of you would know that $R[x]$ is a module over $R$!

Now, let's look at the polynomial rings $R[x]$ and $R[y]$ - they're essentially the same except one consists of polynomials in the variable $x$, and the other in $y$. How are the two related to the polynomial ring $R[x, y]$?

To make things easier, let's fix $R$ to be our favourite ring $\mathbb{Z}$. If I pick an element in $R[x]$, say $x^3 + 5$, and an element from $R[y]$, say $2y$, and multiply the two, then I get the polynomial $(x^3 + 5)(2y) = 2x^3y + 10y$, which is polynomial in $x$ AND $y$, i.e. an element in $R[x, y]$.

OK - so multiplying an element in $R[x]$ with an element in $R[y]$ seems to give us an element in $R[x, y]$. So maybe we can define a multiplication map:

$$\phi\colon R[x] \times R[y] \to R[x, y],$$

which sends $(f(x), g(y))$ to $f(x) \cdot g(y)$.

Can we dare to hope that $\phi$ is a bijection (or maybe an isomorphism of $R$-modules)?

Well, if you've had any experience in solving differential equations, and therein a technique called *separation of variables*, you'd know this map wouldn't be surjective. Indeed, not every polynomial in two variables $x$ and $y$ can be split into two polynomials of one variable each - one in $x$ and the other in $y$. For instance, $x^2y^2 + y$ can't be split into the product of a polynomial in $x$ and one in $y$.

Is it at least injective? Well, the product of $x$ and $2y$ is the same as the product of $2x$ and $y$, meaning that the pairs $(x, 2y)$ and $(2x, y)$ map to the same element under $\phi$, meaning $\phi$ isn't even injective.

So $\phi$ is neither injective nor surjective - very disappointing for a map so promising. Is there ANYTHING nice about $\phi$?

Let's look at what stopped $\phi$ from being a bijection.

1. $\phi$ is not **surjective** because $R[x, y]$ has polynomials other than those that are a product of a polynomial in $x$ and one in $y$. However, these nice

polynomials in the image of $\phi$ that look like $f(x)g(y)$ GENERATE all the polynomials in $R[x,y]$ i.e. any polynomial in two variables can be written as a finite sum of polynomials of the form $f(x) \cdot g(y)$. For instance, in the example from above, $x^2y^2 + y = (x^2)(y^2) + (1)(y)$.

2. $\phi$ is not **injective** because $\phi(2x, y) = \phi(x, 2y)$. In fact, we know exactly when two pairs of polynomials will map to the same thing under $\phi$: for any $c \in R$,
$$\phi(cf(x), g(y)) = \phi(f(x), cg(y)) \tag{1}$$
.

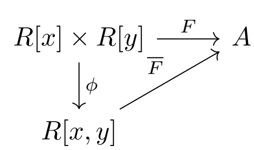
In addition, thanks to the distributive property of polynomials multiplication, we have $\phi(f_1(x) + f_2(x), g(y)) = \phi(f_1(x), g(y)) + \phi(f_2(x), g(y))$, and a similar "linearity" in the second component. This, along with the condition described above in Equation 1 make the map $\phi$ into what's called an *R-bilinear* map. These properties are what make $\phi$ VERY special as we shall see now!

Suppose we had another bilinear map $F\colon R[x] \times R[y] \to A$, where $A$ is any $R$-module, or even just an abelian group. Let us try to define a group homomorphism $\overline{F}\colon R[x,y] \to A$ using $F$.

As observed above, $R[x,y]$ is generated by polynomials of the form $f(x)g(y)$, and so it is sufficient to describe where these generators are mapped to under $\overline{F}$ to completely determine the map. We can therefore define $\overline{F}(f(x)g(y)) = F(f(x), g(y))$. The product $f(x)g(y)$ is of course equal to $\phi(f(x), g(y))$, and therefore we have $\overline{F}(\phi(f(x), g(y))) = F(f(x), g(y))$.

The properties of $\phi$ noted above make sure that this map is well-defined i.e. even though the definition of $\overline{F}$ depends on how you split a polynomial - say $2xy$ - as a product, it won't send the same element in $R[x,y]$ to two different elements in $A$ based on how you split it. For instance, if you write $2xy$ as $2x \cdot y$, then $\overline{F}$ would send it to $F(2x, y)$, and if you split it as $x \cdot 2y$, then $\overline{F}$ would send it to the element $F(x, 2y)$ in $A$. But since $\phi$ and $F$ are bilinear, these two elements in $A$ are equal, so there's no confusion (check this)!

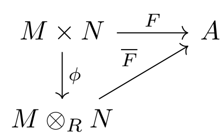
This gives rise to the following commutative diagram (i.e. if you pick an element in $R[x] \times R[y]$, it'll land on the same element in $A$ no matter which path you take) -




Since this holds for *any* choice of abelian group $A$ with a map $F\colon R[x] \times R[y] \to A$, the above property of $R[x,y]$ is called a *universal property*. This group $R[x,y]$ becomes special because it turns bilinear maps from $R[x] \times R[y]$ into group homomorphisms from $R[x, y]$. We call $R[x, y]$ the **tensor product** of $R[x]$ and $R[y]$ and denote it by $R[x] \otimes_R R[y]$ (the subscript $R$ refers to the fact that it is elements $c \in R$ that can be moved around between the two components in the map $\phi$ as in Equation 1) .

We are now (hopefully) motivated enough to define the tensor product of two objects- say modules over a commutative ring $R$ -

Let $M$ and $N$ be two modules over $R$. The tensor product $M \otimes_R N$ of $M$ and $N$ over $R$ is an abelian group with an $R$-bilinear map $\phi \colon M \times N \to M \otimes_R N$ which satisfies the following universal property - for any abelian group $A$ and an $R$-bilinear map $F \colon M \times N \to A$, there exists a unique group homomorphism $\overline{F} \colon M \otimes_R N \to R$ such that the following diagram commutes:




***Bonus Question:*** The whole discussion started with the observation that the product of a polynomial $f(x)$ in $x$ with a polynomial $g(y)$ in $y$ gives a polynomial in two variables $x$ and $y$ i.e. an element in the set $R[x, y]$. But we could have also taken the SUM of $f(x)$ and $g(y)$ and still ended up with an element in $R[x, y]$. Could this have been used to get to the tensor product? Or does the sum map give us some other universal property?