How to ACTUALLY work with Algebraic Groups

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This is going to be a longer post, but hopefully extremely helpful if you ever want to work with algebraic groups.

A birational property of varieties that I study is called R-equivalence. It measures to what extent two rational points of a variety can be joined using rational curves. I will not go into too much detail about R-equivalence, but suffice it to say that "nice" varieties are R-trivial i.e. any two points in the variety can be joined using a rational curve. This includes the affine spaces $\mathbb{A}^n$ and our favorite matrix groups like GL_n and SL_n.

Being an algebraist, I like to specifically investigate the occurrence of this property among (affine) algebraic groups: varieties which have a compatible group structure on them. This extra group structure gives us some reduction techniques to study R-equivalence, and the goal of my blog post is to explain some of these reduction techniques, using R-equivalence as a running example.

Discalimer: throughout, my algebraic groups are smooth and affine.

So let's start with an arbitrary group G over a field k. The R-equivalence class of a point $g \in G(k)$ is the set of all points in $G(k)$ which can be joined to g using a rational curve (you can also just think of this as all the rational curves in G that pass through g). We would like to study the set $G(k)/R$ of R-equivalence classes of G over k.

Step 1: The Connected Components. Suppose $G_1, \ldots, G_n$ are the connected components of G. There are only finitely many of them, and they are all translates of the identity component G^0 (the connected component containing the group identity element e) by elements of G. Any rational curve in G has to lie inside one of these components. The set of R-equivalence classes of G can therefore be determined by computing the set of R-equivalence classes of each of these components, which in turn can be determined by knowing the R-equivalence class set of G^0. Oh and just ignore the component that don't have

any k-points. Therefore, we have reduced the problem of computing the R-equivalence classes of an arbitrary group to the same problem for connected groups.

Step 2: Unipotent and Reductive groups. Every linear algebraic group G can be realized as a closed subgroup of GL_n for some n by some $\overline{k}$-embedding ι. A subgroup H of G is said to be *unipotent* if $\iota(H)$ sits inside the group of unitary matrices inside GL_n (these are the upper triangular matrices with 1's along the diagonals). Yes, there are multiple choices of the embedding ι, and this property will hold for all of them if it holds for one.

Every element in a unipotent group is a *unipotent element* i.e. $\iota(u-e)^m$ is the zero matrix for some m.

We denote by $R_u(G)$ the maximal connected normal unipotent subgroup of G (there is indeed a unique maximal one). We say G is reductive if $R_u(G) = \{e\}$.

We will try to break down an arbitrary connected k-group G into a unipotent part and a reductive part (non-unipotent part). This can be done if the group allows a *Levi decomposition* i.e. if G can be written as a semidirect product $R_u(G) \rtimes H$, where H is a reductive subgroup of G. Unfortunately, it is not always possible to have a Levi decomposition, but it happens frequently enough, for instance whenever $\operatorname{char} k = 0$.

A semidirect decomposition $G \simeq R_u(G) \rtimes H$ means that this is a direct product on the underlying varieties, meaning that the birational geometry of the product can be studied by understanding the birational geometry of the factors. We will use this idea frequently for studying birational geometry. Therefore, to study arbitrary connected k-group, it suffices to understand unipotent k-groups and reductive k-groups. In my particular instance of R-equivalence, we would in this case have $G(k)/R \simeq R_u(G)(k)/R \times H(k)/R$.

Step 3a: Unipotent Groups. We defined unipotent groups using unitary matrices, so let's see what's so special about these matrices. In $GL_2(k)$, the unitary matrices are the 2×2 matrices of the form $M_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, where $x \in k$. This gives a k-isomorphism from the set of unitary matrices to the additive group $\mathbb{G}_a$ by $M_x \mapsto x$. The additive group $\mathbb{G}_a$, defined by $\mathbb{G}_a(A) = A$ for any k-algebra A, is the most fundamental algebraic group. Its underlying variety is simply the affine line $\mathbb{A}^1_k$.

In general, unipotent groups can be thought of being built up from $\mathbb{G}_a$ in the following sense: every unipotent group U admits a *central normal series*, all of whose quotients are isomorphic to $\mathbb{G}_a$. When k is a perfect field, for instance

when $\operatorname{char} k = 0$, we have something even better: the underlying variety of U is actually isomorphic to an affine space $\mathbb{A}^m$. In this case, we therefore have $U(k)/R \simeq \mathbb{A}^m_k(k)/R = 1$. This means that any two k-points in U can be joined using a rational curve.

Over an imperfect field, there's a funny thing that can happen: a unipotent group can look like affine space after extending the field and still have no nonconstant k-maps from $\mathbb{A}^1_k$. Such a group is called *wound*. However, for R-equivalence, we only want *rational* curves, meaning k-maps defined on a nonempty open subset of the affine line, and not necessarily on all of it, so we shouldn't rush into making assertions about R-equivalence just yet!

Step 3b: Reductive Groups. Now that we've dealt with the unitary part of the group, we are left with the reductive part H. Let DH be the commutator algebraic subgroup of H: this is the algebraic subgroup of H generated by elements of the form $h_1 h_2 h_1^{-1} h_2^{-1}$. When H is commutative, DH is of course trivial. The k-group DH turns out to be a *semisimple group*, meaning, that in a sense it is built up from simple groups (see below for more details). The quotient H/DH turns out to also be a special kind of group known as a *k-torus*. A k-group T is said to be a torus if over the algebraic closure $\overline{k}$, T becomes isomorphic to a direct product of copies of the multiplicative group $\mathbb{G}_m = GL_1$. Just like how unipotent groups are built up from copies of $\mathbb{G}_a$, tori are built up from copies of $\mathbb{G}_m$ (see more details below).

So we've found two simpler groups: the semisimple group DH and the torus H/DH. But this time we don't generally get a product. The picture is a little more complicated than that, and I won't get into it in this post.

The R-classes of reductive groups can however still be studied by understanding the geometry of semisimple groups and of tori, but the explanation can get a little technical. I refer the interested reader to Gille's lectures on R-equivalence on algebraic groups.

Step 4a: Algebraic Tori. Like I said before, a k-torus T is built up from copies of $\mathbb{G}_{m,k}$. More precisely, $T(\overline{k}) \simeq \prod_{i=1}^{\ell} \mathbb{G}_{m,k}(\overline{k})$. If this isomorphism is defined over k itself, we say T is a *split torus*. In this case, T inherits a lot of geometric properties from $\mathbb{G}_m$. In particular, $T(k)/R = 1$.

Even when T is not split, we have some tools at our disposal. Every torus T fits into a short exact sequence of k-tori

$$1 \to F \to Q \to T \to 1$$

known as a *flasque resolution* of T. Here, F is a so-called *flasque torus*, which, vaguely speaking, means that it has nice cohomological properties. The

torus Q falls into a class of extremely well-behaved tori known as *quasitrivial tori*. Among other nice properties, Q is k-rational. Flasque resolutions shed some light into the geometry of tori, for instance one can show that $T(k)/R \simeq H^1(k, F)$. As a consequence, when k is a number field, $T(k)/R$ is always finite.

Step 4b: Semisimple groups. An algebraic group is said to be (almost) *simple* if it has no nontrivial (connected) subgroups. As you might have guessed by the name, these, in a sense, are the building blocks of all algebraic groups. More on this later.

A semisimple group G is an *almost direct product* of simple groups, meaning that there is an isogeny $\phi : G_1 \times \cdots \times G_m \to G$, where the G_i are simple groups. (Recall that an isogeny is a surjective morphism with finite kernel.) Therefore, one can study the structure of semisimple groups by understanding simple groups.

Step 5: Simple Groups. We are finally at the final step of our reduction: simple groups. We study simple groups by assigning to each group a connected directed graph, known as its *Dynkin diagram*. The number of vertices of this graph is determined by the rank of a maximal k-split torus S of G. The rule for edges between two vertices in this graph is a little bit more complicated, and I will save it for a future dedicated post on Dynkin diagrams.

A Dynkin diagram (and its variations) can tell us quite a bit about the algebraic group G it came from:

1. if the group G were simply connected (meaning there is no nontrivial isogeny with this group as a codomain) over an algebraically closed field, then the Dynkin diagram completely determines the group. For instance, if the graph of the Dynkin diagram looks like a straight line with n vertices, then the group has to be SL_{n+1}.

2. if the group G weren't simply connected, then the simply connected group G^{sc} that the diagram describes is called the *simply connected cover* of G. This means that there is a isogeny $G^{sc} \to G$, with the finite kernel lying in the center of G^{sc}.

3. the symmetries of the Dynkin diagram (automorphisms on vertices preserving the corresponding edges with multiplicities and directions) correspond to outer automorphisms of the group in the sense that $\mathrm{Sym}(\Delta) \simeq \mathrm{Aut}(G)/\mathrm{Inn}(G)$. Here, Δ denotes the Dynkin diagram, and $\mathrm{Inn}(G)$ denotes the group of inner automorphisms of G, i.e. automorphisms defined using conjugation by an element of G.

4. with some extra structure on the graph, the Dynkin diagram of a group G can help distinguish k-forms of G i.e. k-groups that become isomorphic to G over $\overline{k}$. For instance, if $\mathbb{H}$ is the usual quaternion algebra over $\mathbb{R}$, then $SL_1(\mathbb{H})$ is isomorphic to SL_2 over $\overline{k} = \mathbb{C}$ but not over $\mathbb{R}$ itself. It is therefore a nontrivial k-form of SL_2.

The goal of the next post will be to do a deep dive into Dynkin diagrams and how they keep showing up in all things algebraic group. Ok bye!