Seven Sketches in Compositionality - Exercise 2.73.1

written by FRC on Functor Network
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Problem

Show that a skeletal dagger **Cost**-category is an extended metric space.

Solution

We already know that a **Cost**-category is a Lawvere metric space. We only need to prove that the distance function d of such metric space is symmetric and that $d(x, y) = 0$ implies $x = y$.

The first condition is proved by appealing to the dagger condition. Since $Id : \mathcal{X} \to \mathcal{X}^{op}$ is a **Cost**-functor, $\mathcal{X}(x, y) \geq \mathcal{X}^{op}(x, y) = \mathcal{X}(y, x)$. Dually, $\mathcal{X}(y, x) \geq \mathcal{X}(x, y)$. Thus, since the ordering on the extended real is a total order, we can deduce that $\mathcal{X}(y, x) = \mathcal{X}(x, y)$. This is true for all objects x and y in $\mathrm{Ob}(\mathcal{X})$.

The second condition is proved by appealing to skeletality. Suppose $d(x, y) = 0$. Symmetry tells us that $d(y, x) = 0$. Then, $0 \geq d(x, y)$ and $0 \geq d(y, x)$. Then, $x = y$ by the skeletality condition.