Durrett 1.2

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1.2

1.2.1

For any $B \in \mathcal{B}$, $X^{-1}(B), Y^{-1}(B) \in \mathcal{F}$. Then

$$\begin{aligned}
Z^{-1}(B) &= \{\omega \in \Omega \mid Z(\omega) \in B\} && (1)\\
&= \{\omega \in \Omega \mid (\omega \in A \wedge \omega \in X^{-1}(B)) \vee (\omega \notin A \wedge \omega \in Y^{-1}(B))\} && (2)\\
&= (A \cap X^{-1}(B)) \cup (A^c \cap Y^{-1}(B)) \in \mathcal{F} && (3)
\end{aligned}$$

1.2.2

lower bound: $(2\pi)^{-1/2} \cdot (4^{-1} - 4^{-3}) \cdot \exp(-4^2/2)$

upper bound: $(2\pi)^{-1/2} \cdot (4^{1}) \cdot \exp(-4^2/2)$

1.2.3

For each $x \in (-\infty, \infty)$ set $\delta_x = \lim_{y \to x+} F(y) - \lim_{y \to x-} F(y) = P(\{\omega \in \Omega \mid X(\omega) = x\})$.

Set $D = \{x \in (-\infty, \infty) \mid \delta_x > 0\}$. Then, by the definition above,

$$\sum_{x \in D} \delta_x \leq P(\Omega) = 1$$

A sum of uncountable positive values is infinite, so D must be at most countable.

1.2.4

Set $G : \mathbb{R} \to (0, 1)$ to be the distribution function of $Y = F \circ X$. Then for $y \in \mathbb{R}$, $G(y) = P(Y^{-1}((-\infty, y]))) = P(\{\omega \in \Omega \mid F(X(\omega)) \leq y\})$

Since F maps $\mathbb{R}$ into $[0, 1]$, for $y < 0, G(y) = P(\emptyset) = 0$ and for $y > 1, G(y) = P(\Omega) = 1$.

Set $A = \{\omega \in \Omega \mid F(X(\omega)) \leq y\}$.

Next, since

- F is continuous,

- $\lim_{x\to\infty} F(x) = 1$,
- $\lim_{x\to-\infty} F(x) = 0$,
- $y \in (0, 1)$

Then there must exist at least one x s.t. $F(x) = y$.

- Since F is non-decreasing, the set of all such x are an interval - then set $x* = \sup\{x \in \mathbb{R} \mid F(x) \leq y\}$.
- Since F is continuous, there cannot be a discontinuity at $x*$, so $F(x*) = P(X^{-1}((-\infty, x*])) = y$.

Set $B = X^{-1}((-\infty, x*]) = \{\omega \in \Omega \mid X(\omega) \leq x*\}$.

- Since F is non-decreasing, then for $\omega \in \Omega$, $X(\omega) \leq x* \implies F(X(\omega)) \leq F(x*) = y$
- Since $x* = \sup\{x \in \mathbb{R} \mid F(x) \leq y\}$, $F(X(\omega)) \leq y \implies X(\omega) \leq x*$
- So $A = B$ and $P(A) = G(y) = y$

1.2.5

From the definition of density, set

$$F(x) = \int_{(-\infty, x])} f \, d\lambda$$

Then for $Y = g \circ X$ with distribution function G:

$$\begin{aligned}
G(y) &= P(Y^{-1}((-\infty, y])) && (4)\\
&= P(\{\omega \in \Omega \mid g(X(\omega)) \leq y\}) && (5)\\
&= P(\{\omega \in \Omega \mid X(\omega) \leq g^{-1}(y)\}) && (6)\\
&= F(g^{-1}(y)) && (7)\\
&= \int_{(-\infty, g^{-1}(y)])} f \, d\lambda && (8)
\end{aligned}$$

Since $P(Y^{-1}(\alpha, \beta)) = 1$,

- $\forall y \leq \alpha F(y) = 0$
- $\forall y \geq \beta F(y) = 1$

Then from (7),

- $\forall y \leq g(\alpha), G(y) = F(g(y)) \leq F(g(g^{-1}(\alpha))) = 0$
- $\forall y \geq g(\alpha), G(y) = F(g(y)) \geq F(g(g^{-1}(\beta))) = 1$

Next define the push-forward measure ν given by $\forall B \in \mathcal{B}, \nu(B) = \lambda(g^{-1}(B))$. Then using (i) change of variables and (ii) change of measures:

$$\begin{aligned}
G(y) &= \int_{(-\infty, g^{-1}(y)])} f \, d\lambda && (9)\\
&= \int_{g((-\infty, g^{-1}(y)]))} f \circ g^{-1}, d\nu && (10)\\
&= \int_{(-\infty, y])} f \circ g^{-1}, d\nu && (11)\\
&= \int_{(-\infty, y])} (f \circ g^{-1}) \cdot \frac{d\nu}{d\lambda}, d\lambda && (12)\\
&= \int_{(-\infty, y])} (f \circ g^{-1}) \cdot \frac{1}{g' \circ g^{-1}}, d\lambda && (13)
\end{aligned}$$

1.2.6

The density function of X is

$$f(x) = \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}x^2}$$

Then the density function of $\exp(X)$ is

$$f(x) = \frac{1}{x(2\pi)^{1/2}} e^{-\frac{1}{2}(\ln\, x)^2}$$

1.2.7

$$\begin{aligned}
F_{X^2}(y) &= P(\{\omega \in \Omega \mid X^2(\omega) \in (-\infty, y]\}) && (14)\\
&= P(\{\omega \in \Omega \mid X(\omega) \in [-\sqrt{y}, \sqrt{y}]\}) && (15)\\
&= \int_{[-\sqrt{y}, \sqrt{y}]} f, d\lambda && (16)\\
&= \int_{(-\infty, \sqrt{y}]} f, d\lambda - \int_{(-\infty, -\sqrt{y}]} f, d\lambda && (17)
\end{aligned}$$

Then differentiating:

$$\begin{aligned}
f_{X^2}(y) &= f(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f(-\sqrt{y}) \cdot -\frac{1}{2\sqrt{y}} && (18)\\
&= \frac{f(\sqrt{y}) + f(-\sqrt{y})}{2\sqrt{y}} && (19)
\end{aligned}$$

Then for $X \sim \text{Normal}(0, 1)$,

$$f_{X^2}(x) = \frac{e^{-\frac{1}{2}x}}{(2\pi)^{1/2} \cdot \sqrt{x}}$$