

Durrett 1.2

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1.2

1.2.1

For any $B \in \mathcal{B}$, $X^{-1}(B), Y^{-1}(B) \in \mathcal{F}$. Then

$$Z^{-1}(B) = \{\omega \in \Omega \mid Z(\omega) \in B\} \quad (1)$$

$$= \{\omega \in \Omega \mid (\omega \in A \wedge \omega \in X^{-1}(B)) \vee (\omega \notin A \wedge \omega \in Y^{-1}(B))\} \quad (2)$$

$$= (A \cap X^{-1}(B)) \cup (A^c \cap Y^{-1}(B)) \in \mathcal{F} \quad (3)$$

1.2.2

lower bound: $(2\pi)^{-1/2} \cdot (4^{-1} - 4^{-3}) \cdot \exp(-4^2/2)$

upper bound: $(2\pi)^{-1/2} \cdot (4^1) \cdot \exp(-4^2/2)$

1.2.3

For each $x \in (-\infty, \infty)$ set

$$\delta_x = \lim_{y \rightarrow x+} F(y) - \lim_{y \rightarrow x-} F(y) = P(\{\omega \in \Omega \mid X(\omega) = x\}).$$

Set $D = \{x \in (-\infty, \infty) \mid \delta_x > 0\}$. Then, by the definition above,

$$\sum_{x \in D} \delta_x \leq P(\Omega) = 1$$

A sum of uncountable positive values is infinite, so D must be at most countable.

1.2.4

Set $G : \mathbb{R} \rightarrow (0, 1)$ to be the distribution function of $Y = F \circ X$. Then for $y \in \mathbb{R}$, $G(y) = P(Y^{-1}((-\infty, y])) = P(\{\omega \in \Omega \mid F(X(\omega)) \leq y\})$

Since F maps \mathbb{R} into $[0, 1]$, for $y < 0$, $G(y) = P(\emptyset) = 0$ and for $y > 1$, $G(y) = P(\Omega) = 1$.

Set $A = \{\omega \in \Omega \mid F(X(\omega)) \leq y\}$.

Next, since

- F is continuous,
- $\lim_{x \rightarrow \infty} F(x) = 1$,
- $\lim_{x \rightarrow -\infty} F(x) = 0$,
- $y \in (0, 1)$

Then there must exist at least one x s.t. $F(x) = y$.

- Since F is non-decreasing, the set of all such x are an interval - then set $x^* = \sup\{x \in \mathbb{R} \mid F(x) \leq y\}$.
- Since F is continuous, there cannot be a discontinuity at x^* , so $F(x^*) = P(X^{-1}((-\infty, x^*])) = y$.

Set $B = X^{-1}((-\infty, x^*]) = \{\omega \in \Omega \mid X(\omega) \leq x^*\}$.

- Since F is non-decreasing, then for $\omega \in \Omega$, $X(\omega) \leq x^* \implies F(X(\omega)) \leq F(x^*) = y$
- Since $x^* = \sup\{x \in \mathbb{R} \mid F(x) \leq y\}$, $F(X(\omega)) \leq y \implies X(\omega) \leq x^*$
- So $A = B$ and $P(A) = G(y) = y$

1.2.5

From the definition of density, set

$$F(x) = \int_{(-\infty, x]} f d\lambda$$

Then for $Y = g \circ X$ with distribution function G :

$$G(y) = P(Y^{-1}((-\infty, y])) \tag{4}$$

$$= P(\{\omega \in \Omega \mid g(X(\omega)) \leq y\}) \tag{5}$$

$$= P(\{\omega \in \Omega \mid X(\omega) \leq g^{-1}(y)\}) \tag{6}$$

$$= F(g^{-1}(y)) \tag{7}$$

$$= \int_{(-\infty, g^{-1}(y)]} f d\lambda \tag{8}$$

Since $P(Y^{-1}(\alpha, \beta)) = 1$,

- $\forall y \leq \alpha F(y) = 0$
- $\forall y \geq \beta F(y) = 1$

Then from (7),

- $\forall y \leq g(\alpha), G(y) = F(g(y)) \leq F(g(g^{-1}(\alpha))) = 0$
- $\forall y \geq g(\alpha), G(y) = F(g(y)) \geq F(g(g^{-1}(\beta))) = 1$

Next define the push-forward measure ν given by $\forall B \in \mathcal{B}, \nu(B) = \lambda(g^{-1}(B))$.
Then using (i) change of variables and (ii) change of measures:

$$G(y) = \int_{(-\infty, g^{-1}(y)]} f d\lambda \quad (9)$$

$$= \int_{g((-\infty, g^{-1}(y)]))} f \circ g^{-1}, d\nu \quad (10)$$

$$= \int_{(-\infty, y]} f \circ g^{-1}, d\nu \quad (11)$$

$$= \int_{(-\infty, y]} (f \circ g^{-1}) \cdot \frac{d\nu}{d\lambda}, d\lambda \quad (12)$$

$$= \int_{(-\infty, y]} (f \circ g^{-1}) \cdot \frac{1}{g' \circ g^{-1}}, d\lambda \quad (13)$$

1.2.6

The density function of X is

$$f(x) = \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}x^2}$$

Then the density function of $\exp(X)$ is

$$f(x) = \frac{1}{x(2\pi)^{1/2}} e^{-\frac{1}{2}(\ln x)^2}$$

1.2.7

$$F_{X^2}(y) = P(\{\omega \in \Omega \mid X^2(\omega) \in (-\infty, y]\}) \quad (14)$$

$$= P(\{\omega \in \Omega \mid X(\omega) \in [-\sqrt{y}, \sqrt{y}]\}) \quad (15)$$

$$= \int_{[-\sqrt{y}, \sqrt{y}]} f, d\lambda \quad (16)$$

$$= \int_{(-\infty, \sqrt{y}]} f, d\lambda - \int_{(-\infty, -\sqrt{y}]} f, d\lambda \quad (17)$$

Then differentiating:

$$f_{X^2}(y) = f(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f(-\sqrt{y}) \cdot -\frac{1}{2\sqrt{y}} \quad (18)$$

$$= \frac{f(\sqrt{y}) + f(-\sqrt{y})}{2\sqrt{y}} \quad (19)$$

Then for $X \sim \text{Normal}(0, 1)$,

$$f_{X^2}(x) = \frac{e^{-\frac{1}{2}x}}{(2\pi)^{1/2} \cdot \sqrt{x}}$$