Fun with Van Kampem

Igor Caetano • 31 Dec 2023

Today I want to explain a little about Van Kampen's theorem. First things first, what is it?

Mathematicians often want to have a concrete strategy to distinguish geometric object and shapes from each other. A fundamental tool for such a task is the fundamental group π_1, the fundamental group is quite a simple and intuitive notion, given some points x in our space X then the fundamental groups are simply all the loops starting and ending at x under some equivalence relation. Namely, we say that to paths are the same if they are pointed homotopic to one another, which intuitively just means that we can deform every loop in the equivalence class to each other.

Now, Van Kampen states that given two path connected open sets U, V with a point $x \in U \cap V$ that cover X then $\pi_1(X, x)$ will canonically be the push out of $\pi(U, x) \leftarrow \pi(U \cap V, x) \rightarrow \pi(V, x)$ where $\pi(U \cap V, x) \rightarrow \pi(V, x)$ and $\pi(U \cap V, x) \rightarrow \pi(U, x)$ are the homomorphisms induced by the canonical inclusion $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$.