Discrete Deformations of Shifting Piecewise Sets

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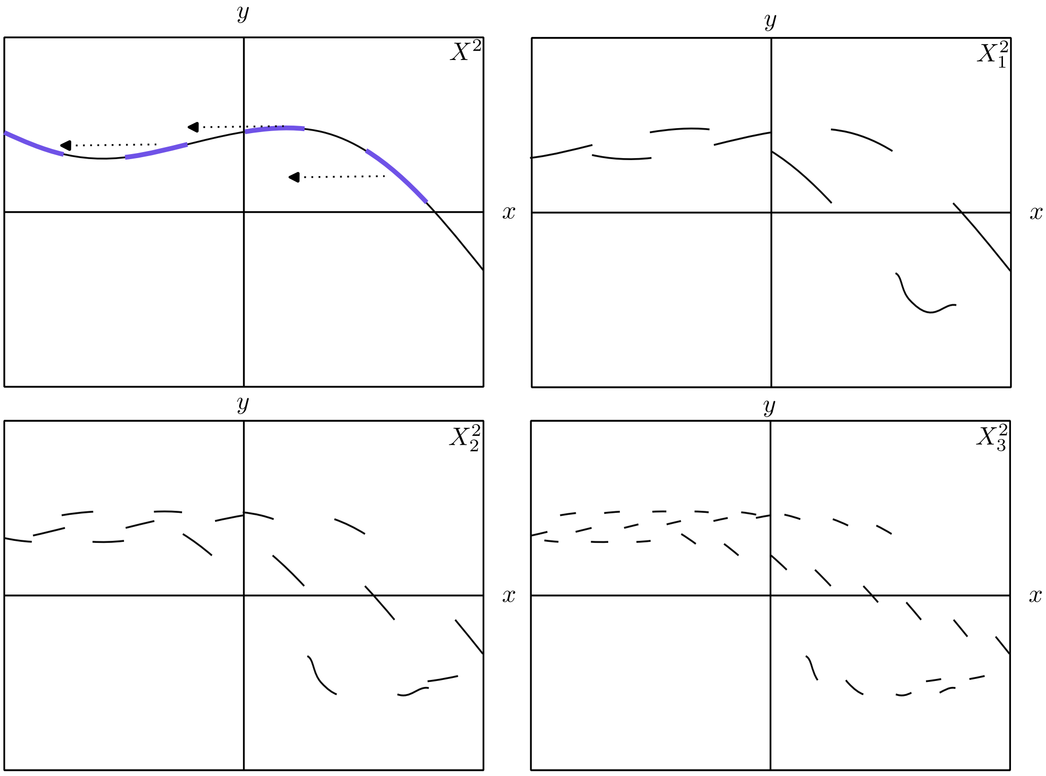
1. Shifting Piecewise Sets

Shifting Piecewise Sets introduce a captivating paradigm where sets, in Euclidean n - space, undergo a dynamic transformation through a sequence of rigid motions(shifts). Imagine a curve embedded in space, systematically skipping and shifting regions along its domain. This process, executed k times on diminishing sized intervals, gives rise to a fascinating array of "distorted" sets. The following visual attempts to illustrate this concept:



This post aims to present the mathematics of how these shifts work, and investigate some of their general implications. I intend to keep things rather informal, exploring what I have uncovered so far. I hope to add on to this post, as I continue to work on this project. Feedback on my writing, mathematics, and other helpful insights would be greatly appreciated.

1.1. Definitions

Definition 1.1.1.

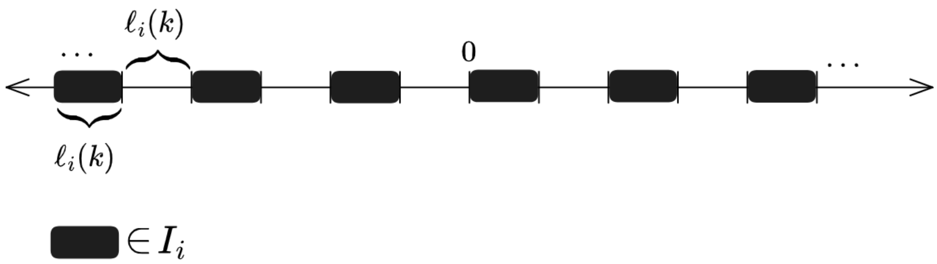
Define the *Extent Encapsulation Vector* to be $\vec{\ell}(k) : \mathbb{N}^n \to \mathbb{R}^n$, and *Extent Parameter* at i to be $\ell_i(k) : \mathbb{N} \to \mathbb{R}$ such that

$$\vec{\ell}(k) = \langle \ell_1(k), \ell_2(k), \dots, \ell_n(k) \rangle.$$

$\ell_i(k)$ denotes a function related to 'length,' dependent on k.

Definition 1.1.2. Let $I_i \subseteq \mathbb{R}$ be defined by

$$I_i = \bigcup_{z \in \mathbb{Z}} [2z \cdot \ell_i(k), (2z+1) \cdot \ell_i(k)].$$



The *Transfer Confinement Space,* $\mathcal{S}_k^n \subseteq \mathbb{R}^n$, at k is denoted,

$$\mathcal{S}_k^n = \bigtimes_{i=1}^{n} I_i.$$

$\mathcal{S}_k^n$ is effectively the Cartesian product of <u>all</u> intervals, I_i, specified by their respective *Extent Parameter,* $\ell_i(k)$.

Definition 1.1.3. Define a function, $\psi_k : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, such that

$$\psi_k(x_1, \dots, x_n, x_{n+1}) = \begin{cases} (x_1 - 2 \cdot \ell_1(k), \dots, x_n - 2 \cdot \ell_n(k), x_{n+1}) \\ \text{ if } (x_1, \dots, x_n) \in \mathcal{S}_k^n. \\ (x_1, \dots, x_n, x_{n+1}) \\ \text{ if } (x_1, \dots, x_n) \notin \mathcal{S}_k^n. \end{cases}$$

We call ψ_k the *Shift Modifier* at k. If an element, $X \in \mathbb{R}^{n+1}$ lies in the *Transfer Confinement Space,* $\mathcal{S}_k^n$ that point gets shifted, as specified by ψ_k.

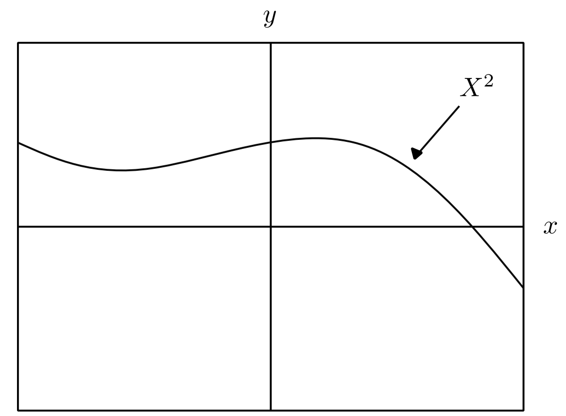
Definition 1.14. We define a function $\Psi_k : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, such that for a given $X \in \mathbb{R}^{n+1}$,

$$\Psi_k(X) = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_2 \circ \psi_1(X).$$

We call Ψ_k the *Spacial Modification Function* for k shifts.

2. First Approach with Curve Shifts

The previous definitions will be applied to the *seed* $X^2 \subseteq \mathbb{R}^2$, a smooth, connected injection with no holes.

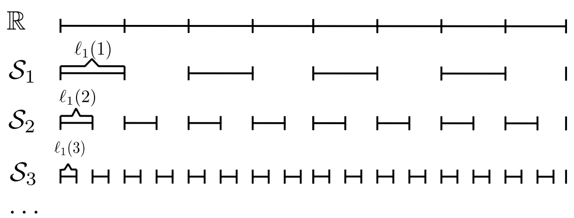


A function, $\ell : \mathbb{N} \to \mathbb{R}$, could be defined such that $\ell(k)$ is strictly decreasing at a rate faster than $\frac{1}{k}$, and $\lim_{n\to\infty} \ell(k) = 0$.

Given ℓ, construct the *Transfer Confinement Space*, at k, $\mathcal{S}_k$ such that,

$$\mathcal{S}_k = \{x \in \mathbb{R} : (\forall z \in \mathbb{Z})(2z \cdot \ell(k) \leq x \leq (2z+1) \cdot \ell(k))\}.$$

Each variation of $\mathcal{S}_k$ contains a set of smaller, closely condensed intervals, as k increases.



The *Shift Modifier*, at k, $\psi_k : \mathbb{R}^2 \to \mathbb{R}^2$, can now be defined.

$$\psi_k(x, y) = \begin{cases} (x - 2 \cdot \ell(k), y) \text{ if } x \in \mathcal{S}_k \\ (x, y) \text{ if } x \notin \mathcal{S}_k. \end{cases}$$

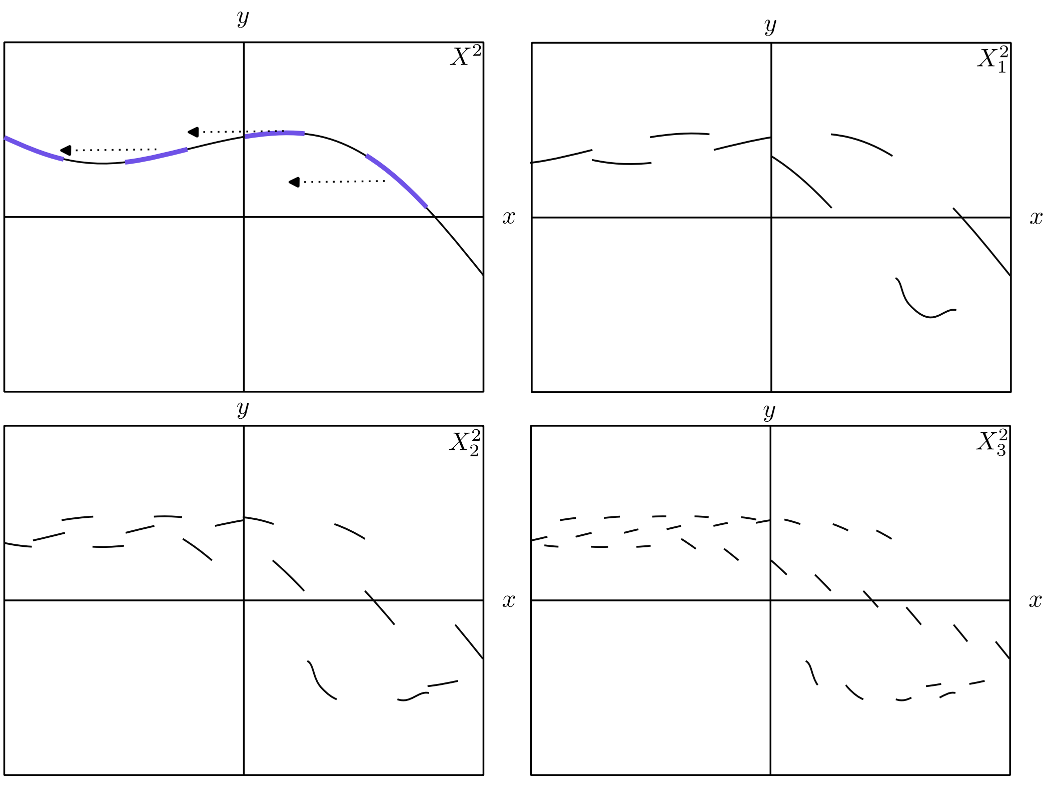
ψ_k relates an element $(x,y) \in \mathbb{R}^2$ to $\mathcal{S}_k$, such that if the x component lies in the contained intervals, specified by $\mathcal{S}_k$, ψ_k shifts the entire coordinate by two lengths of $\ell(k)$ to the left. Otherwise, (x,y) is unchanged.

The *Spacial Modification Function*, $\Psi_k : \mathbb{R}^2 \to \mathbb{R}^2$ is constructed from elements in $\{\psi_1, \psi_2, \ldots, \psi_k\}$, such that

$$\Psi_k(x,y) = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_2 \circ \psi_1(x,y)$$

Ψ_k maps a point, $(x,y) \in \mathbb{R}^2$ to a new position, by changing the value of x, for k shifts. Considering X^2, we can define a different mapping of Ψ_k, for each k, by $X_k^2 \subseteq \mathbb{R}^2$, such that

$$\begin{aligned} X_1^2 &= \{\Psi_1(x,y) : (x,y) \in X^2\} \\ X_2^2 &= \{\Psi_2(x,y) : (x,y) \in X^2\} \\ X_3^2 &= \{\Psi_3(x,y) : (x,y) \in X^2\}. \end{aligned}$$

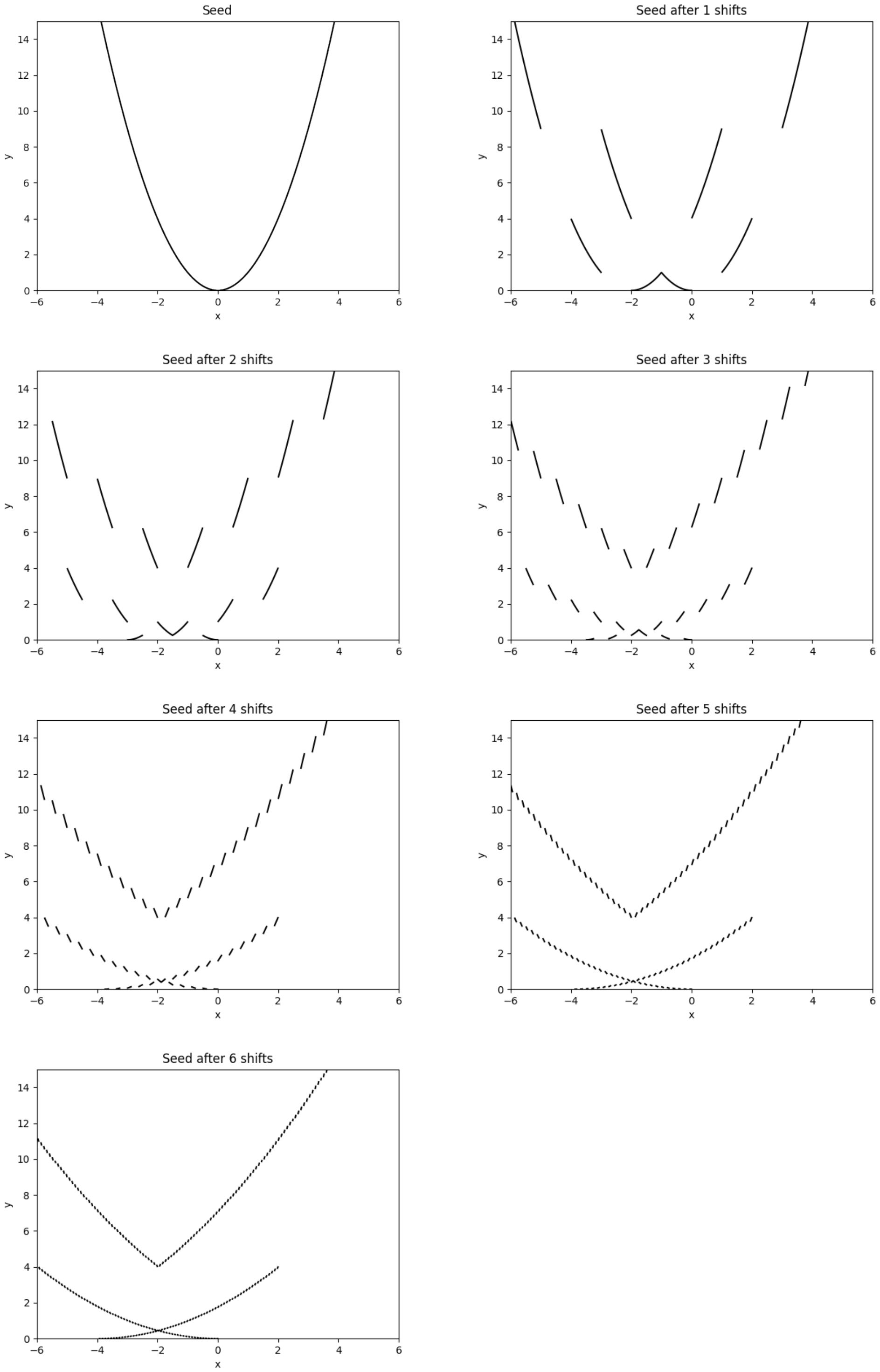


3. Transformations on the Parabola

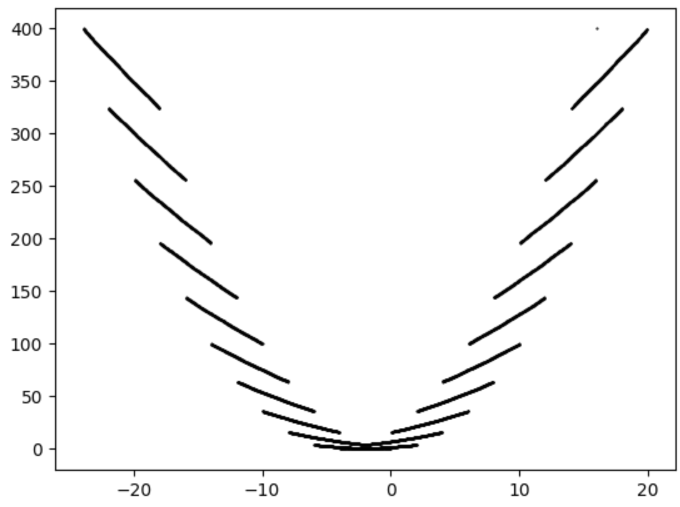
Now that a consensus has been established on what this shifting algorithm does, some examples are in order. What follows are some observations, when modifying the set containing the parabola.

3.1

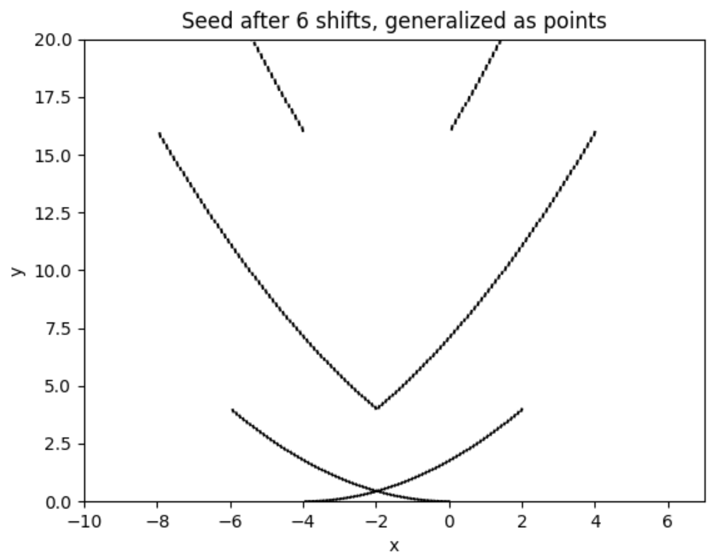
Define the *seed* to be $X^2 = \{(x, x^2) : x \in \mathbb{R}\}$, and *Extent Parameter,* on x to be $\ell : \mathbb{N} \to \mathbb{R}$, such that $\ell(k) = 2^{-(k-1)}$. The following are visual representations of $X^2, X_1^2, \ldots, X_6^2$, respectively.



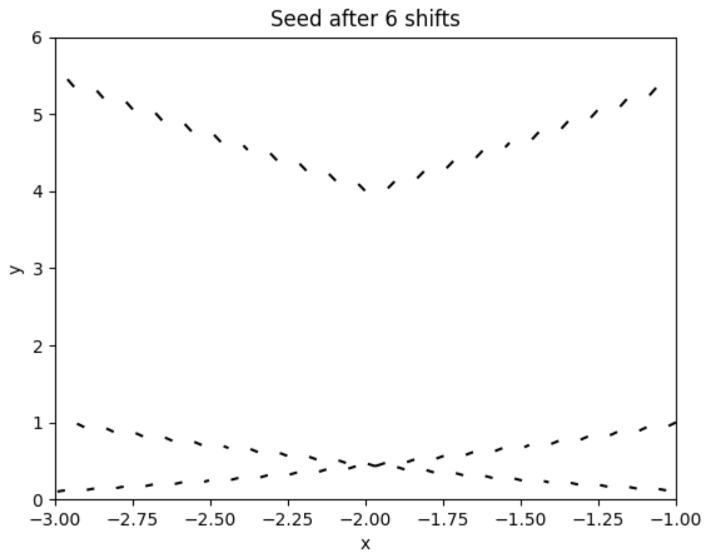
After reorienting this image, and zooming out we observe the following object:



There would appear to be an approaching symmetry about the point $x = -2$.

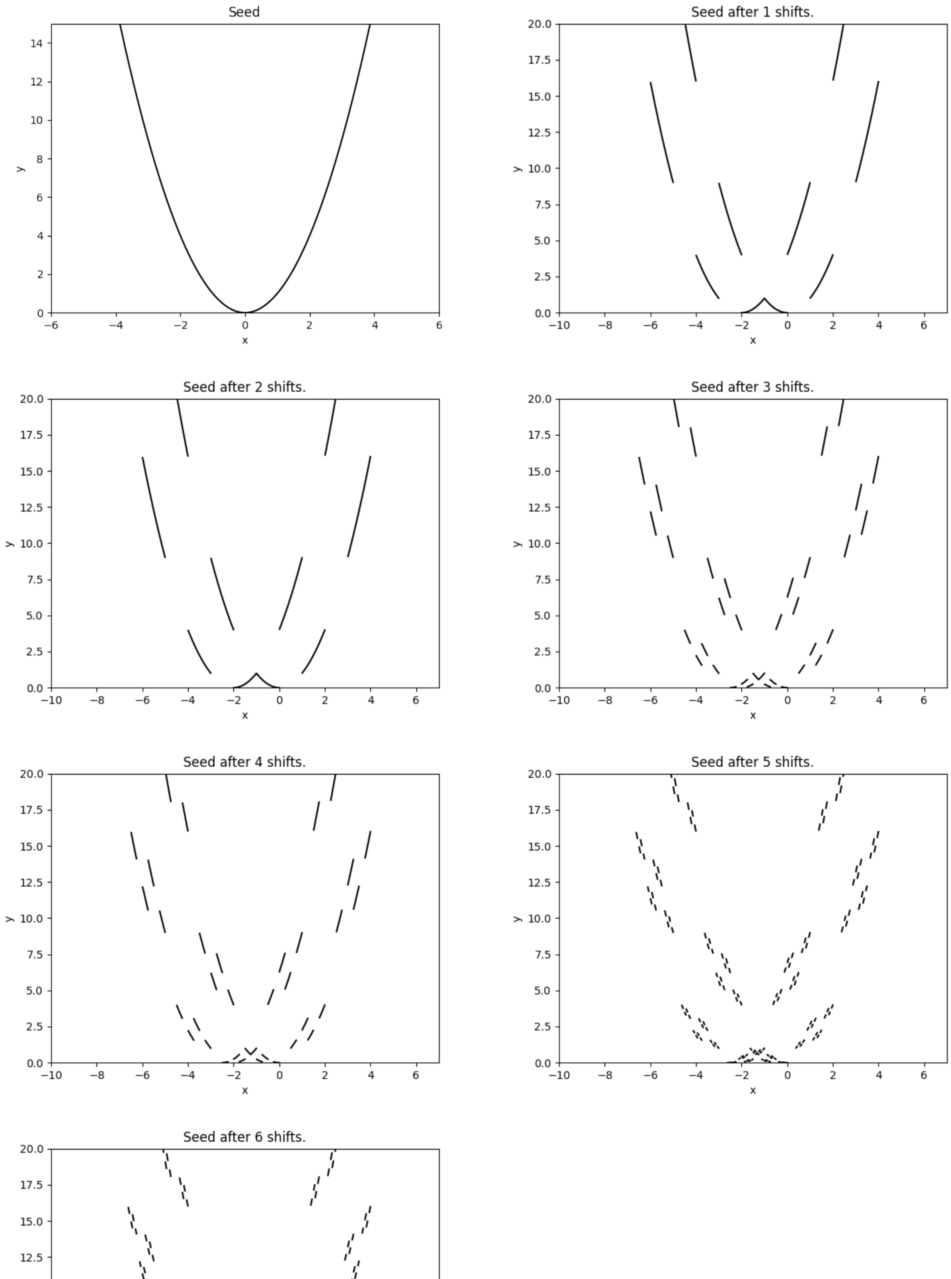


Furthermore, one can see that the resulting set is a bijection from $\mathbb{R} \to \mathbb{R}^2$.

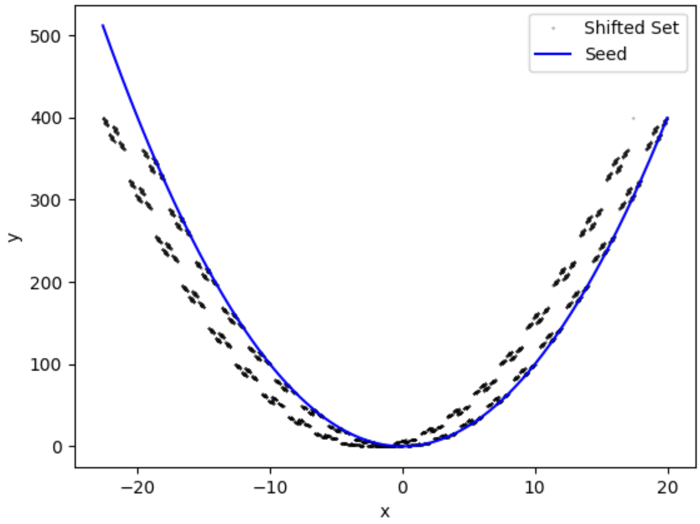


3.2

Define the same *seed*, but change the *Extent Parameter,* on x to be $\ell(k) = 2^{-k}\cos(\pi k)$. The following are visual representations of $X^2, X_1^2, \ldots, X_6^2$, respectively.



Zooming out reveals the following:



The final image maintains some of the structural symmetry observed in the seed.