Vector Sum and Minkowski Sum

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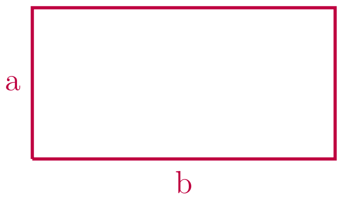
Theorem 1 (*Brunn – Minkowski* inequality for convex bodies II). *Let K and L be two convex bodies (i.e. compact convex sets with nonempty interiors) in $\mathbb{R}^n$, then:*

$$V(K+L)^{1/n} > V(K)^{1/n} + V(L)^{1/n} \tag{1}$$

In geometry, the term $Minkowski\ sum$ is more frequently used for the vector sum. We now give an example to further clarify this claim.
Consider the origin-symmetric rectangle H of sides a and b and the other origin-symmetric rectangle S of sides c and d and their $Minkowski\ sum$ shown below.



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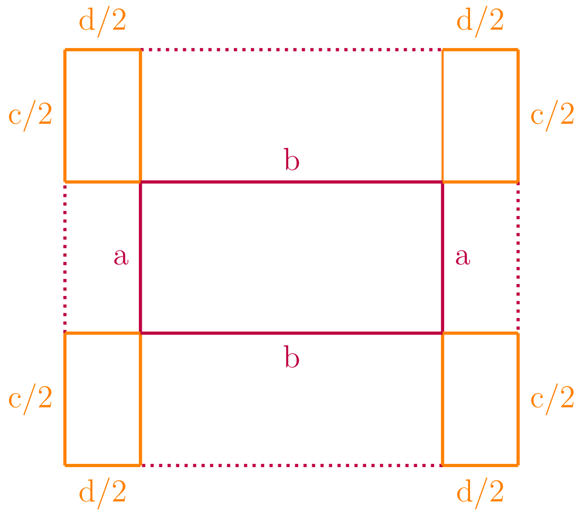

Now we compute their vector sum to show how it is equivalent to the *Minkowski sum* $H+S$:

$$V(H+S) = V(H) + V(S) + 2a(\frac{d}{2}) + 2b(\frac{c}{2}) \tag{2}$$

$$= V(H) + V(S) + ad + bc \tag{3}$$

$$\geq V(H) + 2\sqrt{V(H)V(S)} + V(S) \tag{4}$$

$$= (V(H)^{\frac{1}{2}} + V(S)^{\frac{1}{2}})^2 \tag{5}$$

So we get:

$$V(H+S)^{\frac{1}{2}} \geq V(H)^{\frac{1}{2}} + V(S)^{\frac{1}{2}} \tag{6}$$

which is $Minkowski$ inequality (1) for $n = 2$.

Remark. Notice that we managed to derive equation (8) from equation (7) in the following way:

$$\begin{aligned}
(ad+bc)^2 &\geq 0 \\
(ad)^2 + 2abcd + (bc)^2 &\geq 0 \\
a^2d^2 + b^2c^2 &\geq 2abcd \\
a^2d^2 + 2abcd + b^2c^2 &\geq 4abcd \\
\frac{(ad)^2 + 2abcd + (bc)^2}{4} &\geq abcd \\
(\frac{ad+bc}{2})^2 &\geq V(H)V(S) \\
ad + bc &\geq 2\sqrt{V(H)V(S)}
\end{aligned} \tag{7}$$