Brunn-Minkowski Inequality and the Basic Isomperimetric Inequality

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Theorem 1 (General Brunn-Minkowski inequality in $\mathbb{R}^n$)**.** *Let $0 < \lambda < 1$ and let X and Y be nonempty bounded measurable sets in $\mathbb{R}^n$ such that $(1-\lambda)X + \lambda Y$ is also measurable. Then,*

$$\mu((1-\lambda)X + \lambda Y)^{\frac{1}{n}} \geq (1-\lambda)\mu(X)^{\frac{1}{n}} + \lambda\mu(Y)^{\frac{1}{n}} \tag{1}$$

where μ is the Lebesgue measure.

Remark. We note that in general the assumption that the set $(1-\lambda)X + \lambda Y$ is measurable is necessary even given that both X and Y are measurable. However, for borel sets X and Y, we get that $(1-\lambda)X + \lambda Y$ is a continuous image of their product and so guaranteed being measurable.

Now we show how the $Brunn - Minkowski$ theorem directly implies the very well-know isomperimetric inequlaity:

Corollary 1.1 (The Isoperimetric Inequality). Consider the convex body K with surface area $S(K)$.
Then, the quantity $C(X) = \frac{\mu(K)^{1/n}}{S(K)^{1/(n-1)}}$ is maximized on Euclidean balls.

Proof. By $Brunn - Minkowski$ inequality, we have

$$\begin{aligned}\mu(K+\epsilon B) &\geq (\mu(K)^{1/n} + \epsilon\mu(B)^{1/n})^n \\ &= \mu(K)(1+\epsilon(\frac{\mu(B)}{\mu(K)})^{1/n})^n \\ &\geq \mu(K)(1+n\epsilon(\frac{\mu(B)}{\mu(K)})^{1/n})\end{aligned} \tag{2}$$

where we derived the last inequality by $(1+x)^n \geq 1+nx$ for $x \geq 0$. Now we lower bound the surface area

$$S(K) \geq n\mu(K)(\frac{\mu(B)}{\mu(K)})^{1/n} \tag{3}$$

Now since $S(K) = n\mu(B)$ (by $Minkowski - Steiner$ formula), we get:

$$\begin{aligned}\frac{S(K)}{S(B)} = \frac{S(K)}{n\mu(B)} &\geq \frac{\mu(K)(\frac{\mu(B)}{\mu(K)})^{1/n}}{\mu(B)} \\ &= \mu(K)^{\frac{n-1}{n}}\mu(B)^{\frac{1-n}{n}}.\end{aligned} \tag{4}$$

Finally, we get the isoperimetric inequality:

$$\frac{\mu(B)^{1/n}}{(S(K))^{1/n-1}} \geq \frac{\mu(K)^{1/n}}{S(K)^{1/n-1}} \tag{5}$$

□

Corollary 1.2 (The Classical Isoperimetric Inequality in the Plane).

$$L^2 \geq 4\pi A \tag{6}$$

where A is the area of a domain enclosed by a curve of length L.

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