

Brunn-Minkowski Inequality and the Basic Isoperimetric Inequality

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Theorem 1 (General Brunn-Minkowski inequality in \mathbb{R}^n). *Let $0 < \lambda < 1$ and let X and Y be nonempty bounded measurable sets in \mathbb{R}^n such that $(1 - \lambda)X + \lambda Y$ is also measurable. Then,*

$$\mu((1 - \lambda)X + \lambda Y)^{\frac{1}{n}} \geq (1 - \lambda)\mu(X)^{\frac{1}{n}} + \lambda\mu(Y)^{\frac{1}{n}} \quad (1)$$

where μ is the Lebesgue measure.

Remark. We note that in general the assumption that the set $(1 - \lambda)X + \lambda Y$ is measurable is necessary even given that both X and Y are measurable. However, for borel sets X and Y , we get that $(1 - \lambda)X + \lambda Y$ is a continuous image of their product and so guaranteed being measurable.

Now we show how the *Brunn – Minkowski* theorem directly implies the very well-know isoperimetric inequalaity:

Corollary 1.1 (The Isoperimetric Inequality). Consider the convex body K with surface area $S(K)$.

Then, the quantity $C(K) = \frac{\mu(K)^{1/n}}{S(K)^{1/(n-1)}}$ is maximized on Euclidean balls.

Proof. By *Brunn – Minkowski* inequality, we have

$$\begin{aligned} \mu(K + \epsilon B) &\geq (\mu(K)^{1/n} + \epsilon\mu(B)^{1/n})^n \\ &= \mu(K)(1 + \epsilon(\frac{\mu(B)}{\mu(K)})^{1/n})^n \\ &\geq \mu(K)(1 + n\epsilon(\frac{\mu(B)}{\mu(K)})^{1/n}) \end{aligned} \quad (2)$$

where we derived the last inequality by $(1 + x)^n \geq 1 + nx$ for $x \geq 0$. Now we lower bound the surface area

$$S(K) \geq n\mu(K)(\frac{\mu(B)}{\mu(K)})^{1/n} \quad (3)$$

Now since $S(K) = n\mu(B)$ (by *Minkowski – Steiner* formula), we get:

$$\begin{aligned} \frac{S(K)}{S(B)} &= \frac{S(K)}{n\mu(B)} \geq \frac{\mu(K)(\frac{\mu(B)}{\mu(K)})^{1/n}}{\mu(B)} \\ &= \mu(K)^{\frac{n-1}{n}} \mu(B)^{\frac{1-n}{n}}. \end{aligned} \quad (4)$$

Finally, we get the isoperimetric inequality:

$$\frac{\mu(B)^{1/n}}{(S(K))^{1/n-1}} \geq \frac{\mu(K)^{1/n}}{S(K)^{1/n-1}} \quad (5)$$

□

Corollary 1.2 (The Classical Isoperimetric Inequality in the Plane).

$$L^2 \geq 4\pi A \quad (6)$$

where A is the area of a domain enclosed by a curve of length L .

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