Prekopa-Leindler inequality

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Theorem (Prekopa-Leindler inequality (an integral form of the Brunn-Minkowski inequality)) Let $0 < \lambda < 1$ and f, g, and h be nonnengative integrable function on $\mathbb{R}^n$ such that

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda} \tag{1}$$

for all $x, y \in \mathbb{R}^n$. Then, we have:

$$\int_{\mathbb{R}^n} h(x)dx \geq (\int_{\mathbb{R}^n})^{1-\lambda} (\int_{\mathbb{R}^n} g(x)dx)^{\lambda} \tag{2}$$

Proof. We assume that A, B and $A+B$ are measurable sets, and we shift them such that $A \cap B = \phi$, and so

$$A + B \geq A \cup B \tag{3}$$

and so by almost disjointedness we get

$$\mu(A+B) \geq \mu(A) + \mu(B) \tag{4}$$

Now we show the inequality by induction:

1. Basic Step $n = 1$: Define the set

$$L_h(t) = x : h(x) \geq t \tag{5}$$

   and similarly,

$$L_f(t) = x : f(x) \geq t \tag{6}$$

$$L_g(t) = y : g(y) \geq t \tag{7}$$

   From 3, we get

$$L_h(t) \supseteq \lambda L_f(t) + (1-\lambda) L_g(t) \tag{8}$$

   and so from 4, we get

$$\mu(L_h(t)) \geq \mu(\lambda L_f(t)) + \mu((1-\lambda) L_g(t)) \tag{9}$$

   and since we are only considering the one dimensional (n=1) case, we get

$$\mu(L_h(t)) \geq \lambda \mu(L_f(t)) + (1-\lambda)\mu(L_g(t)) \tag{10}$$

   Without loss of generality, assume that $f \geq 0$, then by Fubini's theorem we get

$$\int_{\mathbb{R}} h(x)dx = \int_{t \geq 0} \mu(L_h(t))dt \tag{11}$$

$$\begin{aligned} \int_{\mathbb{R}} h(x)dx &= \int_{t \geq 0} \mu(L_h(t))dt \\ &\geq \int_{t \geq 0} (\lambda \mu(L_f(t)) + (1-\lambda)\mu(L_g(t)))dt \\ &= \lambda \int_{t \geq 0} \mu(L_f(t))dt + (1-\lambda) \int_{t \geq 0} \mu(L_g(t))dt \\ &= \lambda \int_{\mathbb{R}} f(x)dx + (1-\lambda) \int_{\mathbb{R}} g(y)dy \end{aligned} \tag{12}$$

   Now, from (**??**, below), we get that

$$\int_{\mathbb{R}} h(x)dx \geq (\int_{\mathbb{R}} f(x)dx)^{\lambda} (\int_{\mathbb{R}} g(x)dx)^{1-\lambda} \tag{13}$$

2. Inductive Step: Suppose the inequality holds for $\mathbb{R}^n$, now we show it also holds true for $\mathbb{R}^{n+1}$:
   Let $x, y \in \mathbb{R}^n$, and $\alpha, \beta \in \mathbb{R}$
   Set

$$\gamma = \lambda\alpha + (1-\lambda)\beta \tag{14}$$

   Define the following function for any constant c:

$$h_c(x) = h(c, x) \tag{15}$$