Prekopa-Leindler inequality

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Theorem (Prekopa-Leindler inequality (an integral form of the Brunn-Minkowski inequality)) Let $0 < \lambda < 1$ and f, g, and h be nonnengative integrable function on $\mathbb{R}^n$ such that

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda} \tag{1}$$

for all $x, y \in \mathbb{R}^n$. Then, we have:

$$\int_{\mathbb{R}^n} h(x)dx \geq (\int_{\mathbb{R}^n})^{1-\lambda}(\int_{\mathbb{R}^n} g(x)dx)^{\lambda} \tag{2}$$

Proof. We assume that A, B and $A+B$ are measurable sets, and we shift them such that $A \cap B = \phi$, and so

$$A + B \geq A \cup B \tag{3}$$

and so by almost disjointedness we get

$$\mu(A+B) \geq \mu(A) + \mu(B) \tag{4}$$

Now we show the inequality by induction:

1. Basic Step $n = 1$: Define the set

   $$L_h(t) = x : h(x) \geq t \tag{5}$$

   and similarly,

   $$L_f(t) = x : f(x) \geq t \tag{6}$$

   $$L_g(t) = y : g(y) \geq t \tag{7}$$

   From 3, we get

   $$L_h(t) \supseteq \lambda L_f(t) + (1-\lambda)L_g(t) \tag{8}$$

   and so from 4, we get

   $$\mu(L_h(t)) \geq \mu(\lambda L_f(t)) + \mu((1-\lambda)L_g(t)) \tag{9}$$

   and since we are only considering the one dimensional (n=1) case, we get

   $$\mu(L_h(t)) \geq \lambda\mu(L_f(t)) + (1-\lambda)\mu(L_g(t)) \tag{10}$$

   Without loss of generality, assume that $f \geq 0$, then by Fubini's theorem we get

   $$\int_{\mathbb{R}} h(x)dx = \int_{t \geq 0} \mu(L_h(t))dt \tag{11}$$

$$\begin{aligned}
\int_{\mathbb{R}} h(x)dx &= \int_{t\geq 0} \mu(L_h(t))dt \\
&\geq \int_{t\geq 0} (\lambda\mu(L_f(t)) + (1-\lambda)\mu(L_g(t)))dt \\
&= \lambda \int_{t\geq 0} \mu(L_f(t))dt + (1-\lambda)\int_{t\geq 0} \mu(L_g(t))dt \\
&= \lambda \int_{\mathbb{R}} f(x)dx + (1-\lambda)\int_{\mathbb{R}} g(y)dy
\end{aligned} \tag{12}$$

Now, from (**??**, below), we get that

$$\int_{\mathbb{R}} h(x)dx \geq (\int_{\mathbb{R}} f(x)dx)^{\lambda}(\int_{\mathbb{R}} g(x)dx)^{1-\lambda} \tag{13}$$

2. Inductive Step: Suppose the inequality holds for $\mathbb{R}^n$, now we show it also holds true for $\mathbb{R}^{n+1}$:
   Let $x, y \in \mathbb{R}^n$, and $\alpha, \beta \in \mathbb{R}$
   Set

$$\gamma = \lambda\alpha + (1-\lambda)\beta \tag{14}$$

Define the following function for any constant c:

$$h_c(x) = h(c, x) \tag{15}$$

where $(c, x) \in \mathbb{R}^{n+1}$. Similarly,

$$f_c(x) = f(c, x) \tag{16}$$

$$g_c(y) = g(c, y) \tag{17}$$

Hence, from (14) and (15), we get:

$$\begin{aligned}
h_\gamma(\lambda x + (1-\lambda)y) &= h(\lambda\alpha + (1-\lambda)\beta, \lambda x + (1-\lambda)y) \\
&= h(\lambda(\alpha, x) + (1-\lambda)(\beta, y))
\end{aligned} \tag{18}$$

From (1), we get:

$$h_\gamma(\lambda x + (1-\lambda)y) \geq f(\alpha, x)^{\lambda} g(\beta, y)^{1-\lambda} \tag{19}$$

From (16) and (17), we finally get:

$$h_\gamma(\lambda x + (1-\lambda)y) \geq f_\alpha(x)^{\lambda} g_\beta(y)^{1-\lambda} \tag{20}$$

Recall that the induction hypothesis applied to the function $f_\gamma, f_\alpha, g_\beta$ is

$$\int_{\mathbb{R}^n} h(x)dx \geq (\int_{\mathbb{R}^n} f(x)dx)^{\lambda}(\int_{\mathbb{R}^n} g(x)dx)^{1-\lambda} \tag{21}$$

For simplicity, set

$$\begin{aligned}
H(\gamma) &= \int_{\mathbb{R}^n} h_\gamma(z)dz \\
F(\alpha) &= \int_{\mathbb{R}^n} f_\alpha(z)dz \\
G(\beta) &= \int_{\mathbb{R}^n} g_\beta(z)dz
\end{aligned} \tag{22}$$

We rewrite (21) using (22) and (14)

$$H(\lambda\alpha + (1-\lambda)\beta) \geq F(\alpha)^\alpha G(\beta)^{1-\lambda} \tag{23}$$

Notice that in the basic case (n=1), we proved the statement for any fixed $\alpha, \beta \in \mathbb{R}$, and so the functions H, F, and G satisfy the hypothesis for the one-dimensional case. Thus, we have

$$\int_{\mathbb{R}} H(\gamma)d\gamma \geq (\int_{\mathbb{R}} F(\alpha)d\alpha)^\lambda (\int_{\mathbb{R}} G(\beta)d\beta)^{1-\lambda} \tag{24}$$

Finally, applying Fubini's theorem, we get:

$$\int_{\mathbb{R}^{n+1}} h(x)dx \geq (\int_{\mathbb{R}^{n+1}} f(x)dx)^\lambda (\int_{\mathbb{R}^{n+1}} g(x)dx)^{1-\lambda} \tag{25}$$

Therefore, from steps 1 and 2 and via induction, we conclude that for any $n \in \mathbb{N}$, we have

$$\int_{\mathbb{R}^n} h(x)dx \geq (\int_{\mathbb{R}^n} f(x)dx)^\lambda (\int_{\mathbb{R}^n} g(x)dx)^{1-\lambda} \tag{26}$$

□