Weak metrizability of a ball and separability

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Let E be a Banach space and E' its continuous dual. By B_E we denote the closed unit ball in E centered at zero. We have the following metrization theorem together with its “dual” form:

> Theorem 1. $B_{E'}$ is weak*-metrizable iff E is separable.

> Theorem 2. B_E is weakly metrizable iff E' is separable.

Why these theorems are interesting? Well, an infinite-dimensional Banach space is never weakly metrizable (same for a dual space with weak* topology). But a natural corollary of the above theorems states that all bounded subsets are weakly metrizable whenever dual is separable (and the analogous version for the weak* topology. By the way, Terence Tao made a mistake about this fact which makes the hypothesis that he is only a semi-god more probable 🙃.

In this post we will prove both theorems and discuss alternative proofs. We will kill two birds with one stone since proofs of Theorem 1 and Theorem 2 are quite similar. However, there is one place where this symmetry breaks, which makes the whole feat more interesting!

We mostly follow Brezis' textbook “Functional Analysis, Sobolev Spaces and Partial Differential Equations”. However, the proof of the ⇒ part of Theorem 2 and shorter proofs in the Comments section are mine (I guess that they belong to folklore, let me know if you saw them somewhere).

We jump straight to the proofs. If needed, the reader can find necessary definitions in the last section. But a familiarity with a basic functional analysis is assumed.

Proof of ⇐

Theorem 1: Separable E implies a weak*-metrizable dual ball.

Proof. Let (x_n) be a dense sequence in E. The map

$$B_{E'} \ni f \mapsto \sum_{n=1}^{\infty} \frac{|f(x_n)|}{2^n}$$

defines a norm that agrees with the weak* topology on B'_E. This needs a cumbersome verification which can be found in the mentioned Brezis' textbook.

□

Theorem 2: Separable E' implies a weakly metrizable ball.

Proof. Proceed analogously with a dense sequence (f_n) and the norm $x \mapsto \sum |f_n(x)| \cdot 2^{-n}$.

□

Remark. The reader might have seen a similar series while proving that a countable product of metric spaces remains a metric space. A shorter proof of the ⇐ part of Theorem 1 that relies on this general fact is described in Comments.

Proof of ⇒

Theorem 1. Weak*-metrizable $B_{E'}$ implies separable E.

Proof. Assume that $B_{E'}$ is weak*-metrizable with a metric d.

For each natural number n let

$$B_n = \{f \in B_{E'} : d(0, f) < 1/n\}.$$

Note that these open balls form a local base of zero. Moreover, for each B_n there is a weak* open basic set $U_n \subset B_n$. That is, there is a finite $F_n \subset E$ and $\epsilon_n > 0$ with

$$U_n = \{f \in B_{E'} : |f(x)| < \epsilon_n \text{ for } x \in F_n\}.$$

Now, let

$$F = \bigcup_n F_n.$$

It suffices to show that the linear span of F is dense in E since finite linear combinations of elements of F with rational coefficients $\mathrm{span}_{\mathbb{Q}} F$ form a countable dense subset of $\mathrm{span}\ F$.

To show that the subspace $\mathrm{span}\ F$ is dense in E we use the following fact:

Fact 1. A linear subspace A of a normed space X is dense iif whenever $f \in X'$ disappears on A then f disappears on the whole space.

Take $f \in E'$ with $f|_{\text{span } F} = 0$. Then for all $x \in F$ we have $f(x) = 0$. This means $f \in U_n$ for all n. But since $U_n \subset B_n$, we have

$$f \in \bigcap B_n = \{0\}.$$

And we are done.

□

Theorem 2. If B_E is weak-metrizable then E' is separable.

The proof is by me but it probably belongs to folklore.

Proof. Here the symmetry breaks… That is, we cannot simply rewrite the previous proof exchanging E with E' etc. (as it was the case in the ⇐ part of the proof). Nevertheless, we start somehow similarly, modulo a technical caveat: We need to consider E as a subset of the bidual E'' via the canonical map

$$\Phi \colon E \to E''$$
$$\Phi(x) = \phi_x$$

where $\phi_x(f) = f(x)$. Why we need to do this? Try to proceed without this step and see where the proof breaks. Anyway, the canonical map is an isometry hence

$$\Phi(B_E) \subset B_{E''}.$$

The map Φ is also a homeomorphism between E with the weak topology and $\Phi(E) \subset E''$ with the weak* topology. Therefore $\Phi(B_E) \subset E''$ is metrizable with the subspace topology inherited from $\sigma(E', E)$ with some metric d.

Then $\{B_n = B(0, n^{-1}) \subset (\Phi(B_E), d) : n = 1, 2, \ldots\}$ is a local base of zero. There are weak* open basic sets

$$U_n = \{x \in B_{E''} : |f(x)| < \epsilon_n \text{ for } f \in F'_n\} \subset B_n$$

where $F' \subset E'$ is finite and $\epsilon_n > 0$. Let

$$F' = \bigcup_n F'_n.$$

Take any $T \in E''$ such that the restriction $T|_{\text{span } F'} = 0$. As before, showing that $T = 0$ completes the proof. Now, however, we cannot proceed analogously because T might not live in $\Phi(E)$.

WLOG $T \in B_{E''}$ (otherwise scale T, do the work, and scale it back). Since $\Phi(B_E)$ is weak*-dense in $B_{E''}$ (Goldstine lemma) we find a net (T_α) in $B_{E''}$ with

$$T_\alpha \overset{\sigma(E'', E')}{\to} T.$$

This means that for every $f \in E'$ we have $T_\alpha(f) \to T(f)$. In particular, since F'_n is finite, for ϵ_n there is β_n such that $\alpha > \beta_n$ implies

$$|T_\alpha(f) - \underbrace{T(f)}_{T|_{F'_n}=0}| = |T_\alpha(f)| < \epsilon_n \quad (\forall f \in F'_n).$$

Hence for any B_n there is β_n such that $\alpha > \beta_n$ implies $T_\alpha \in U_n \subset B_n$. This means $T_\alpha \to 0$ in $B \subset B_{E''}$. Recall that $T_\alpha \to T$ by assumption. Since T_α is a net in B and the weak* topology is Hausdorff (this guarantees unique limits), we conclude that $T = 0$ as needed.

□

Comments

An alternative (shorter but less direct) proof of the ⇐ part of Theorem 1 by me (but, again, probably belongs to the folkolore): A separable E implies that $B_{E'}$ is weak*-metrizable.

Proof. Let $D = \{x_n : n = 1, 2, \dots\}$ be a dense subset of E. Consider the following map

$$B_{E'} \ni f \mapsto f|_D \in \mathbb{R}^{\mathbb{N}}.$$

It is injective (functions to a Hausdorff space are uniquely described on a dense subset). It is continuous (in both topologies convergence = pointwise convergence). Since the domain is compact (by Banach–Alaoglu theorem) it must be homeomorphism onto its image.

□

The analogous version of this proof for the “dual” statement, namely the ⇐ part of Theorem 2, has to be modified since B_E does not need to be weakly compact (in fact unit ball B_E is weakly compact iff E is reflexive). So compared to the ⇐ part of Theorem 1 an extra work has to be done (namely, the continuity of the inverse).

However, we can choose another path. Using the already proven Theorem 1 we show that a separable dual of a Banach space implies that a ball is weakly metrizable.

Proof. Assume that E' is separable. Then $B_{E''}$ is weak*-metrizable by Theorem 1. Since B_E with the weak topology can be considered as a subset of $B_{E''}$ with the weak* topology on bidual we are done.

□

Definitions

Let E be a Banach space. By E' we denote a set of continuous linear functionals. By B_E $(B_{E'})$ we denote the closed unit ball (dual) centered at zero.

By the weak topology on E we mean a weak topology generated by the dual E'. We denote it by $\sigma(E, E')$. This is the smallest topology on E that makes all functions from E' continuous. A subbasic set is of the form

$$\{x \in E : |f(x)| < \epsilon\},$$

where $f \in E'$ and $\epsilon > 0$. By the weak* topology on E' we mean the weak topology generated by the set of all evaluation functionals $\{\phi_x : x \in E\}$, that is $\phi_x(f) = f(x)$ for each $f \in E'$. Since they can be identified with elements of E it is sensible to denote the weak* topology by $\sigma(E', E)$. A subbaisc set if of the form

$$\{f \in E' : f(x) < \epsilon\},$$

where $x \in E$ and $\epsilon > 0$.