Sun Set — a connected subset of a plane that is a union of pairwise disjoint closed sets

Kamil'σ math blog • 6 Apr 2024

The classical result of Sierpinski says that a compact connected Hausdorff space cannot be expressed as a countable union of nontrivial pairwise disjoint closed sets.

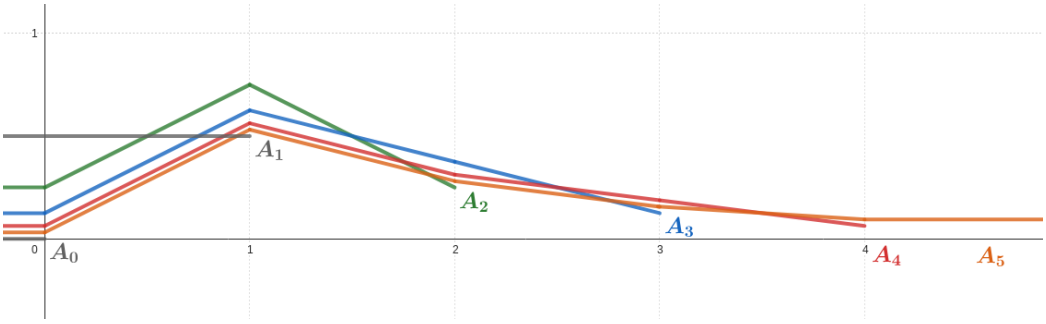
In $\mathbb{R}^n$ this implies that **a closed, bounded and connected subset cannot be decomposed in a countable union of nontrivial pairwise disjoint closed sets**. A natural question arises: *Can we construct a connected subset of a Euclidean space which decomposes into countably many pairwise disjoint closed sets?* I stumbled a few days ago upon this this nice question.

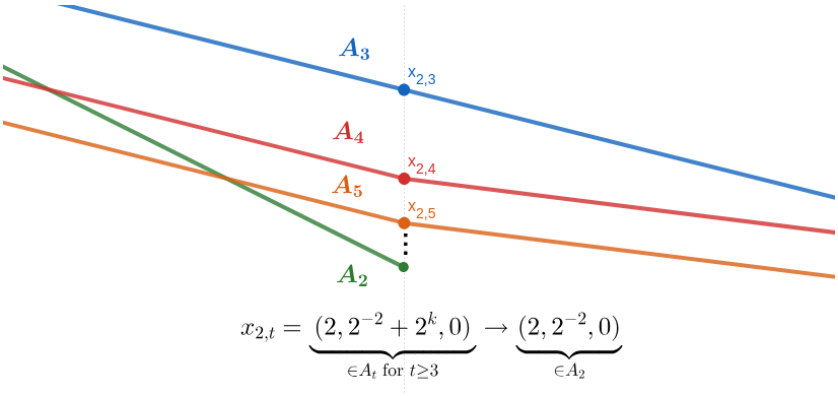
Give this problem a try! For me it was a very nice exercise. I haven't seen it in any textbook, which is quite suprising.

What follows is a short description of my first approach to the problem. This produces an example in $\mathbb{R}^3$ instead of a plane (hence it deserves the name "ugly example"). Next, a differnet and much simpler example is introduced. I call it "Sun set". Post finishes with historical remarks where the last example, Mullikin's nautilus, is shown.

Ugly example

First, I've come up with an example in $\mathbb{R}^3$. Construction is fairly simple but wordy. Here are the pictures of it (third dimension, here depth, is only to make "bridges" to avoid intersections we see in the pictured 2D projection).





Here are the details of this construction, if you are interested in it.

I didn't manage to simplify this construction so that it could live in the plane. It seems like the extra dimension for "bridges" to avoid crossings are necessary. This problem "fights" me, nice. A clever trick or a different approach is needed.

Sun set

I abandoned my original idea and went with a different approach. Here is a description of my example of a connected subset of a plane that can be written as a coutable union of nontrivial pairwise disjoint closed sets.

Let $D = \{d_n : n = 0, 1, \dots\}$ be a dense subset in the unit circle with $d_n \neq d_m$. For each $n = 1, 2, \dots$ define the line segment

$$A_n = \{\lambda d_n : 2^{-n} \leq \lambda \leq 1\}.$$

And let $A_0 = \{\lambda d_0 : 0 \leq \lambda \leq 1\}$. Finally, let $\text{Sun} = \bigcup_n A_n$. It clearly is a union of closed, pairwise disjoint, connected sets.

Theorem. Sun is connected.

Proof. Let U, V be open disjoint sets with $\text{Sun} = U \cup V$. Without loss of generality assume that U contains the origin $0 \in A_0$. Since A_0 is connected, it must be contained in U. It suffices to show that $A_1, A_2, \dots$ live in U as well.

First, note that there is a natural number N for which the ball $B = B(0, 2^{-N}) \cap A$ is contained in the open set U. Consequently, for each $k > N$ the endpoint of A_k, namely $2^{-k}d_k$, is close enough to the origin to be contained in $B \subset U$. Hence, all A_k with $k > N$ are in U (since every A_k is connected).

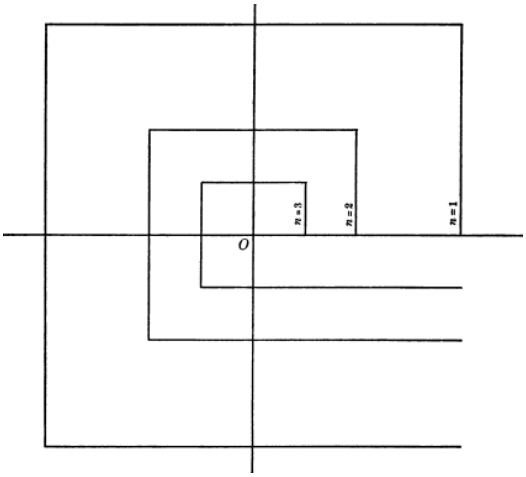
It remains to show that $A_1, A_2, \dots, A_N$ are also in the open set U. Pick any index $1 \leq t \leq N$. By construction $d_t \in A_t$. Aiming for a contradiction, assume that d_t lives in the open set V. Then there is a neighborhood of d_t with no points

in common with U. This is impossible because U contains $\{d_k : k > N\} \subset D$ which is dense in the unit sphere. Hence d_t belongs to U, and so A_t is in U (A_n's are connected).

□

A historical note

Anna Mullikin, in her 1922 paper "Certain Theorems Relating to Plane Connected Point Sets", constructs *Mullikin's nautilus*. It is formed by taking a union of infinitely many polygonal chains. The first three are pictured below (screenshot from the original paper).



So neat! She has done something I couldn't — her example has a similar feel to my first example in $\mathbb{R}^3$. But, with a clever shape, she obtained the needed limit points to "connect" parts without the necessity of my overengineered "3D bridges".

The virtually same set was constructed by Kuratowski (see the second volume of his textbook titled "Topology", or "Continuum theory" by Nadler). I thank my friend Julia who showed me this reference during math student conference OSSM 2024 in Kraków.

Post a comment if you come up with a different construction!