Upper bound topology of T4 topologies might not be T4 (examples)

Kamil'σ math blog • 19 Jan 2024

Given a family of topologies $\{\tau_i : i \in I\}$ on a fixed set X the smallest topology in which any τ_i-open set is open is called the least upper bound topology. It is a standard fact that when each τ_i satisfies a separation axiom T_S, then the supremum topology is also T_S for $S = 0, 1, 2, 3, 3\frac{1}{2}$. However, this cannot be strengthened to T_4. I could not find any explicit examples on the web or in literature, so I constructed some and described them here.

Definitions

Before we go to the details, let us fix the notation.

Def. *Least upper bound topology*. Given a family of topologies $\{\tau_i : i \in I\}$ on a given set X the least upper bound topology τ is generated by the subbase $\bigcup_i \tau_i$. Sometimes we abuse notation and denote the supremum topology by $\bigcup_i \tau_i$.

Def. Topological space X is T_4 when it is T_1 and normal: for two non-trivial, disjoint closed sets E, F there are disjoint open sets U, V that separate them.

Def. *Sorgenfrey line*. Equip set of reals with the topology generated by $\{[a, b) : a, b \in \mathbb{R}\}$. The resulting topology, τ_S, is called the Sorgenfrey line. Note that (a, b) is τ_S-open. Hence, the τ_S is strictly finer (larger) than the topology on Euclidean line.

Sketch of construction

We will construct two T_4 spaces such that the least uper bound topology is not T_4. Now I begin to describe the construction, so this is the last call to try to find one for yourself!

We will first argue that the "half Euclidean, half Sorgenfrey plane", that is, the product of the real line with the Euclidean topology, $(\mathbb{R}, \tau_E)$, and the Sorgenfrey line, $(\mathbb{R}, \tau_S)$ is normal.

Then we will show that the least upper bound topology of $(\mathbb{R}, \tau_E) \times (\mathbb{R}, \tau_S)$ and $(\mathbb{R}, \tau_S) \times (\mathbb{R}, \tau_E)$ is just a Sorgenfrey plane — a classic example of a non-normal space (to see this play with, spoiler alert, the set $\{(x, -x) : x \in \mathbb{R}\}$).

Then I will show a bonus example that generalizes.

The construction

The Sorgenfrey line is not only T_4, but also T_6.

Def. Space X is T_6 when for any two closed, non-trivial disjoint sets E, F there is a continuous function $f : X \to [0,1]$ such that $f^{-1}(\{0\}) = E$ and $f^{-1}(\{1\}) = F$.

To show that the Sorgenfrey line is T_6 we will use the following characterization of T_6 spaces.

Theorem 1 (Vedenissoff) Given a topological space X the following are equivalent:

- X is T_6.
- Every closed set E can be written as a countable intersection of open sets (= G_δ set).

The Sorgenfrey line is T_6 because for any closed set E we have

$$E = \bigcap_n \left(\bigcup_{x \in E} [x, x + n^{-1}) \right).$$

Now, knowing that the Sorgenfrey line is T_6, we can use the next theorem to show that the product of the Sorgenfrey line with the Euclidean line is normal.

Theorem 2 A product of a T_6 space and a metric space is T_6.

For details see Dan Ma's post.

Summing up: Both

$$(\mathbb{R}^2, \tau_{S,E}) := (\mathbb{R}, \tau_S) \times (\mathbb{R}, \tau_E)$$

and

$$(\mathbb{R}^2, \tau_{E,S}) := (\mathbb{R}, \tau_E) \times (\mathbb{R}, \tau_S)$$

are T_6 (the order in which the product is taken does not matter).

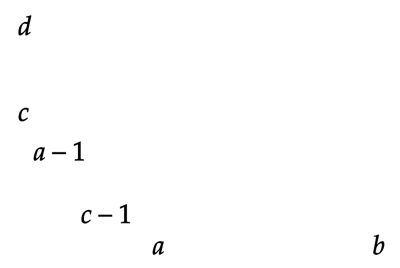
It remains to show that the supremum topology of these two T_4 topologies,

$$\tau := \tau_{E,S} \cup \tau_{S,E},$$

is the same as the topology on the Sorgenfrey plane, τ', which is not T_4.

First note that a basic open set of a Sorgenfrey plane, that is, a rectangle $[a, b) \times [a, c)$, is open in the upper bound topology because we can write it as an intersection of basic open sets:

$$\Big((a-1, b) \times [c, d)\Big) \cap \Big([a, b) \times (c-1, d)\Big).$$



Now the other way around. Let U be any subbasic set from the supremum topology. By definition, U is open in at least one of two topologies $\tau_{E,S}$ or $\tau_{S,E}$. Without loss of generality, we assume that U is open in $\tau_{E,S}$. Hence, there is an index set I such that

$$U = \bigcup_{i \in I, A_i \in \tau_E, B_i \in \tau_S} A_i \times B_i.$$

But $\tau_E < \tau_S$ so each A_i is also τ_S-open. Hence, U is just a union of a product τ_S-open sets. But this is a definition of an open set in the product topology. Therefore, U is open in the Sorgenfrey plane. □

Bonus example

More elementary construction (but using an infinite family of τ_i's) of non-T_4 supremum topology made from T_4 spaces is this: It is known that $\mathbb{N}^{\mathbb{R}}$ is not normal ($\mathbb{N}$ with the discrete topology). We argue that the following supremum topology is not T_4:

$$\tau = \bigcup_{x \in \mathbb{R}} \tau_x,$$

where τ_x consists of the empty set and sets

$$\prod_{i \in \mathbb{R}} \begin{cases} \mathbb{N}, & i \neq x, \\ A \subset \mathbb{N}, A \neq \emptyset, & i = x. \end{cases}$$

Each $(\mathbb{N}^{\mathbb{R}}, \tau_x)$ is T_4 because it is homeomorphic to the discrete space on natural numbers. On the other hand, $(\mathbb{N}^{\mathbb{R}}, \tau)$ is homeomorphic to $\mathbb{N}^{\mathbb{R}}$. □

More generally, since every $T_{3\frac{1}{2}}$ space is embeddable in the product of $[0,1]$, I think, we could make an analogous construction as above and represent *any* $T_{3\frac{1}{2}}$ space as the upper bond topology of a product of T_4 spaces.

Notes

It is easy to create a family of spaces that are normal (closed sets can be separated by open sets, but space might not be T1), but whose supremum topology is not normal. For example, let $X := \{1,2,3\}$. Define three topologies

$$\tau_1 := \{\{1,2\},\emptyset,X\}, \tau_2 := \{\{2\},\emptyset,X\}, \tau_3 := \{\{2,3\},\emptyset,X\}.$$

Each of these topologies is vacuously normal, but

$$\bigcup \tau_i = \{\{2\},\{1,2\},\{2,3\},\emptyset,X\}$$

is not normal.

I think an appropriate way to finish this post is by linking to *This is not normal* by Negativland.