# Box topology is rarely metrizable

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## Summary

In this post, I show that the infinite product of non-discrete space is not metrizable.

There are quite a few questions on math SE about the metrizability of $\mathbb{R}^{\mathbb{R}}$, $\mathbb{R}^{\omega}$, $[0,1]^{[0,1]}$, etc. equipped with the box topology (see here or here for example). For me, those were too specific to see what is the key property of a space used to prove non-metrizability.

## Theorem

Recall that for a topological space $X$ the box topology on the product $\prod_{i\in I} X$ is generated by the basic sets of the form

$$\prod_{i\in I} U_i, \quad \text{where } U_i \subset X \text{ is open.}$$

**Theorem 1 (Box topology is rarely metrizable)** *Let $X$ be a non-discrete topological space. Then infinite product $\prod_{i\in I} X$ is not metrizable.*

*Proof.* We consider two cases.
*Case 1: $X$ is Hausdorff.* We will show that $\prod_i X$ is not first countable, hence not metrizable. Let $x$ be a non-isolated point in $X$ (which exists by the assumption that $X$ is not discrete). For the sake of contradiction assume that $\{ A_n : n \in \mathbb{N} \}$ is coutable at $(x_i)_{i\in I}$ where $x_i = x$ for each $i \in I$. For any $A_n$ we find a basic open neighborhoodwhich contains $(x_i)$, namely

$$A_n \supset \prod_i U_i^{(n)} \ni (x_i),$$

where each $U_i^{(n)}$ is an open neighborhoodof $x_i$. Since $I$ is infinite we take an infinite countable subset $I' \subset I$. For convinience relabel elements of $I$ so that $I' = \{i_1, i_2 \dots\}$.

For any $n \in \mathbb{N}$ let $y^{(n)} \in U_{i_n}^{(n)} \subset X$ be a point distinct from $x$. Such a point exists because we assumed that $x$ is not isolated. Since $X$ is Hausdorff we can separate $x$ from $y^{(n)}$ with an open neighborhood$V_n$ of $x$, so that $y^{(n)} \notin V_n$. Let

$$V = \prod_i \begin{cases} V_n, & \text{for } i = i_n, \\ X, & \text{otherwise.} \end{cases}$$

Set $V$ is a basic open neighborhoodof $(x_i)$. It is enough to show that $A_n \not\subset V$ for any $n \in \mathbb{N}$. Assume that for some natural number $k$ we have $A_k \subset V$. For $i \in I$ let

$$y_i = \begin{cases} y^{(k)}, & \text{for } i = i_k, \\ x, & \text{otherwise.} \end{cases}$$

and define $(y_i) \in \prod_i X$. Clearly $(y_i) \in \prod_i U_i^{(k)} \subset A_k$. On the other hand

$$y_{i_k} = y^{(k)} \notin V_k$$

by construction of $V_k$. Hence $(y_i) \notin V$. We arrive at a contradiction with the inclusion $A_k \subset V$.

*Case 2. $X$ is not Hausdorff.* It is enough to show that $\prod_i X$ with the box topology is not Hausdorff. Let $x, y$ be the points such that for any open neighborhoods $U, V$ of $x, y$ (respectively) we have $U \cap V \neq \emptyset$. Then points $(x_i), (y_i)$ where $x_i = x$ and $y_i = y$ cannot be separated by basic open sets in the box topology. □

# Personal context (it is a blog after all)

Theorem 1.3 in the *Handbook of set-theoretic topology* by Kunen and Vaughan is similar. But there it is assumed that $X$ is infinite, non-discrete, Hausdorff, and completely regular... On top of that proof seemed complicated to me — after skimming it I decided, following advice from Thurston's *On proof and progress in mathematics*, that it would be easier to come up with my own proof. It probably wasn't easier, but for sure it was more fun! If you see any mistakes, please point them out.