Tychonoffication with an application

Kamil'σ math blog • 9 Nov 2023

Summary

In this post, I show how to turn any topological space into a Tychonoff space (which meets a certain universal property). This construction has a neat application in the theory of spaces of real continuous functions equipped with the topology of pointwise convergence, known as C_p-theory, namely: Given space X there is a Tychonoff space X' such that $C_p(X)$ and $C_p(X')$ are isomorphic (homeomorphic and holomorphic).

Recall that when X is a topological space then $C_p(X) = (C(X), \tau_p)$ is the space of real continuous functions equipped with the topology of pointwise convergence τ_p which is generated by a basis formed by

$$[x_1, \dots x_n; U_1, \dots, U_n] \coloneqq \{\, f \in C(X) : f(x_i) \in U_i \text{ for } 1 \le i \le n \,\},$$

where $n \in \mathbb{N}$, each $x_i \in X$ and U_i is open in $\mathbb{R}$. Equivalently, we can view the set of all continuous real functions on X as a subset of all real functions, in symbols $C(X) \subset \mathbb{R}^X$. When we equip $\mathbb{R}^X$ with the product topology, then $C_p(X)$ has simply the subspace topology.

Context

Let's look at the broader context first. I identified two motifs.

Motif 1. X → X'. Given a topological space and a desired property, how to change it in some minimal or optimal way. Compactification is a good example. In our case, the desired properties are separation axioms.

Motif 2: Interplay between X, Cp(X) and Cp(Y). This is fun for the sake of exploring and finding counterexamples. For example, check Drawing Sorgenfrey continuous functions from Dan Ma's Topology Blog.

Our goal is to turn X into X' which is Tychonoff and $C_p(X) \cong C_p(X')$. This greatly simplifies C_p-theory: It allows us to assume that X is Tychnoff while working with $C_p(X)$. See it in action in the theorem below.

Let X be Tychonoff. $C_p(X)$ is metrizable if and only if X is coutable.

Theorem 1 Let X be Tychonoff. $C_p(X)$ is metrizable if and only if X is coutable.

Proof outline $\Rightarrow$ Assume there is a countable base at f. Derive at contradiction using a Cantor-diagonal-argument-like trick to create an open neighborhood of f which is not a supset of any set from a local base. $\Leftarrow$ Note that $C_p(X)$ is a subset of $\mathbb{R}^X$ with product topology, hence metrizable as a countable product of metrizable space. $\square$

$\Rightarrow$ We are proving the contrapositive. Assume that X is uncountable, but $C_p(X)$ is metrizable. Metric spaces are first countable so we can pick a countable local base $\mathcal{A}$ at the zero function θ. For $A \in \mathcal{A}$ take a basic set $B_A \subset A$ which contains θ. Let $\text{support}[x_1, \ldots, x_n; U_1, \ldots, U_n] \coloneqq \{x_1, \ldots, x_n\}$ and

$$Y \coloneqq \bigcup_{A \in \mathcal{A}} \text{support}(B_A) \subset X.$$

Set Y is countable as a countable union of finite sets. Because X is uncountable there is $x \in X \setminus Y$. Since $[x; (-1,1)] \eqqcolon V_x$ is an open neighborhood of θ and $\mathcal{A}$ is a base at that point there is $A \in \mathcal{A}$ such that

$$V_x \supset A \supset B_A = [x_1, \ldots, x_k; U_1, \ldots, U_k].$$

Since X is Tychonoff we find $f \in C(X)$ such that $f(x_i) \in U_i$ for $1 \leq i \leq k$ and $f(x) = 1 \notin (-1, 1)$. The function f is well defined because $x \notin \text{support}(B_A)$. By construction $f \in B_A$ but $f \notin V_x$. Contradiction with $V_x \supset B_A$.

$\Leftarrow$ Let X be countable. Note that $C_p(X) \subset \mathbb{R}^X$. Since $\mathbb{R}^X$ is metrizable as a countable product of metrizable space, space $C_p(X)$ is metrizable as a subset of a metrizable space $\mathbb{R}^X$. $\square$

Hints that we can simplify space X to X’ so that CpX ≃ CpX’

Before we move on to the Tychonoffication let’s wonder a bit.

Recall that that a topological space X is *not Kolmogorov* (not $T0$) when there is a pair of distinct points x and y such that every open set containing one point, contains the other. Any trivial topology on a set with at least two points is not $T0$.

We can think of it as if topology could not detect a difference between x and y. If that is the case, it is a straightforward exercise to show:
$f \in C(X) \implies f(x) = f(y)$.

This *hints to us that we can simplify X by 'gluing' points that are topologically indistinguishable* and expect that spaces of continuous functions will be isomorphic (this is Kolmogorov quotient or Kolmogorofication if you like). Now the question is: Can we simplify X even more?

Consider the following example. Let X be an $\mathbb{R}$ with the excluded point topology with 0 (that is $U \neq X, \emptyset$ is open only when $0 \in U$). This space is $T0$ but the only real continuous functions are constant. Hence $C_p(X) \cong C_p(\{0\})$.

Now we advise you to ponder: Which points in X we can glue so that $C_p(X) \cong C_p(X')$ (where X' is a space with glued points)? The answer is in the next section.

Tychonoffication

We will not glue the points which are topologically indistinguishable, but instead, those points which are indistinguishable by continuous real functions!

Formally we define an equivalence relation

$$x \sim y \iff \forall f \in C(X) \text{ we have } f(x) = f(y).$$

Then topologize $X/\sim$ with the weak topology on a certain family of functions — the quotient topology might be too fine for our purposes (we will provide an example later).

Before the construction, we need a definition and an important lemma.

Definition. Let $\mathcal{F}$ be a family of functions from a set X to a topological space Y. The *weak topology* on X, denoted $\sigma(X, \mathcal{F})$ is the smallest topology which makes functions from $\mathcal{F}$ continuous. Its subbasis consists of sets of the form $f^{-1}(U)$, where $f \in \mathcal{F}$ and $U \subset Y$ is open.

Lemma 2 Let X be a set and $\mathcal{F} \subset \mathbb{R}^X$. Then $(X, \sigma(X, \mathcal{F}))$ is a completely regular space. Moreover, if $\mathcal{F}$ separates points in X, then this space is Tychonoff.

Let A be a nontrivial closed set. Pick x in A^c which is open. Hence there exists a basic neighborhood

$$x \in \bigcap_{i=1}^{n} f_i^{-1}(U_i) \subset A^c,$$

where $f_i \in \mathcal{F}$ and $U_i \subset \mathbb{R}$ is open for each $1 \leq i \leq n$. Since $\mathbb{R}$ is completely regular and point $f_i(x)$ is not in the closed U_i^c, there is a continous function $g_i \colon \mathbb{R} \to [0,1]$ such that $g_i(f_i(x)) = 1$ and $g_i(U_i^c) = \{0\}$. Let

$$h \coloneqq (g_1 \circ f_1) \cdots (g_n \circ f_n).$$

The function $h \colon X \to [0,1]$ is continuous as a finite product of continuous functions. We have $h(x) = 1 \cdots 1 = 1$. Take $a \in A$. There has to be an index i for which $f_i(a) \notin U_i$, since otherwise $a \in \bigcap_{i=1}^n f_i^{-1}(U_i) \subset A^c$ which is a contradiction sine a is in A. Thus $g_i(f_i(a)) = 0$ and $h(a) = 0$. Hence $h(A) = \{0\}$.

Now assume that $\mathcal{F}$ separates points in X. To show that X is Tychonoff it remains to show that X is Hausdorff. Take different points $x, y \in X$. By assumption, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. Since $\mathbb{R}$ is Hausdorff there exist disjoint open neighborhoods U, V of $f(x), f(y)$ respectively. Sets $f^{-1}(U) \ni x$ and $f^{-1}(V) \ni y$ are weakly open sets separating x and y. □

Construction

We outline the construction of Tychonoffication given any topological space. It is straightforward to verify each step.

Step I. Define an equivalence relation:

$$x \sim y \iff \forall f \in C : f(x) = f(y).$$

Step II. For $f \in C(X)$ define

$$\varphi_f = \{([x], f(y)) : [x] \in X/\!\sim \text{ and } y \in [x] \text{ is arbitrary}\}.$$

It's straightfoward to see that φ_f is a function $\varphi_f \colon X/\!\sim \to \mathbb{R}$.

Step III. Construct the Tychonoff topology on $X/\!\sim$. Let $\mathcal{F} = \{\, \varphi_f : f \in C(X) \,\}$. Note that this family of functions separates points in $X/\!\sim$. By lemma the weak topology $\sigma(X/\!\sim, \mathcal{F}) =: \tau$ is Tychonoff.

And that's all!

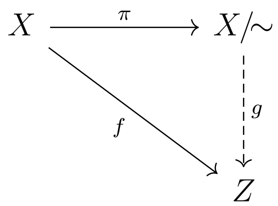
Remark. The weak topology on $X/\!\sim$ from the above theorem may not be the same as a quotient topology τ_q. Since projection π is weakly continuous and τ_q is the finest topology that makes π continuous, we have $\tau \subset \tau_q$. This inclusion might be strict as we show in the following example.

Example Take the K-topology on $\mathbb{R}$. A basis of this topology is formed by open intervals (a, b) and $(a, b) - K$, where $K = \{\, 1/n : n \in \mathbb{Z}_+ \,\}$. Because K-topology is strictly finer than $\tau_{\mathbb{R}}$, the identity function $f \colon (\mathbb{R}, \tau_K) \to (\mathbb{R}, \tau_{\mathbb{R}})$ is continuous. This means that $x \sim y \iff x = y$. So each equivalence class is a singleton. Thus $(\mathbb{R}, \tau_K) \cong (\mathbb{R}/{\sim}, \tau_q)$. But K-topology is not regular. And in particular, it's not Tychonoff. On the other hand $\mathbb{R}/{\sim}$ with the weak topology from the preceding theorem is Tychonoff.

Universal property

The Tychonofication process described is optimal in the sense that it enjoys the following universal property:

Let X be a space and $X/{\sim}$ its Tychonoffication. For any Tychonoff space Z and continuous transformation $f\colon X \to Z$ there is a $unique$ and continuous $g\colon X/{\sim} \to Z$ such that $f = g \circ \pi$. In other words, the following diagram commutes:



Tychonoffication and Cp-theory

Theorem 3 For any space X let $X/\sim$ be its Tychoniffication. Then $C_p(X) \cong C_p(X/_{\sim})$.

Everything that was built so far, is for this proof to be boring. That's why I give only an outline.

Proof outline Show that canonical projection $\pi\colon X \to X/\sim$ is continuous.

Define

$$\varphi\colon C_p(X) \to C_p(X/\sim)$$

$$f \mapsto \varphi_f$$

Show that φ is a bijection.

Moreover, φ is a homomorphism. For any $f, g \in C(X)$ and $[x]$ we have

$$\varphi(f+g)([x]) = (f+g)(x) = f(x) + g(x) = \varphi(f)([x]) + \varphi(g)([x]).$$

Lastly, φ is a homeomorphism. We will only show that φ is continuous. Let $[[x]; U]$ be an arbitrary subbasic element of $C_p(X/{\sim})$ (it is enough to check continuity on subbasic sets). For any $x \in [x]$ the set

$$\varphi^{-1}(U) = \{ f \in C(X) : f(x) \in U \}$$

is open in $C_p(X)$ as standard subbasic element. Hence φ is continuous. $\square$

And we are done! In Tkachuk's book *A Cp-Theory Problem Book: Topological and Function Spaces* all of the exercises numbered 101–500 spaces are assumed to be Tychonoff thanks to this single theorem.

Notes and related concepts

- Most part of this post is written based on a solution to the 100th problem in Tkatchuk's aforementioned book.
- Metrizatoin theorem might be proved in many other ways. For example by noting that $C_p(X)$ is a convex space. See Theorem 4 in *Topological Vector Spaces* by Robertsons. Or by noting that $C_p(X)$ is a topological group and using the Birkfoff-Kakutani theorem. I really liked how it was proved in the *Lectures in Functional Analysis and Operator Theory* (Theorem 6.3) by Berberian.
- Turing X into Kolmogorov space is very easy, into Tychonoff is just a bit laborious. How about other separation axioms? It's quite surprising process of turning X into Hausdorff involves transfinite induction! See the Bachleor's thesis of Bart van Munster The Hausdorff Quotient.

Edits: Fixed typos and added definition of $C_p(X)$.