

CIRCLE TANGENCY CRITERION FOR CYCLIC QUADRILATERAL

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Circle Tangency Criterion for Cyclic Quadrilateral

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Abstract

We present a bidirectional characterization of cyclic quadrilaterals based on of two localized sub circles generated by diagonal intersections, orthocenter,

Introduction

The structural properties of cyclic quadrilaterals form a rich chapter in Euclidean geometry. A particularly elegant section within this domain involves the configurations generated by the intersecting diagonals and sub-triangles they make. Works of Johnson, Altshiller-Court, and others have studied symmetries, collinearity of orthocenter and circumcenter of opposite triangles, and tangency with their shared vertex.

While these classical properties regarding collinearity and concurrency are well known, the direct relationship to internal circle tangency criteria remains largely unexplored.

In this paper, we use the geometric properties of diagonals of cyclic quadrilaterals to derive a new necessary and sufficient condition for the cyclicity of a quadrilateral. We will demonstrate that a convex quadrilateral is cyclic if and only if two uniquely constructed circles achieve mutual tangency at the diagonal intersection point.

2. Notations and Diagram

Let $ABCD$ be a convex quadrilateral, whose diagonals AC and BD intersect at Q .

formalize the following definitions:

- Let O be the circumcenter of the diagonal triangle QCD
 - We define ω_1 as the circle with diameter QO .
 - Midpoint of QC is N , and midpoint of QD is M .
 - Let α denote centre of ω_1 .
- Let H be the orthocentre of the opposite diagonal triangle QAB
 - We define ω_2 as the circle with diameter QH
- Foot of perpendicular from vertex B of triangle QAB onto QA is G , and from vertex A of triangle QAB onto QB is F .
 - Let β denote centre of ω_2 .
- Let I denote foot of altitude from vertex Q of triangle QAB onto AB .
 - ω is circumcircle of triangle QCD .

3. Proof

3.1 Theorem Statement

The Quadrilateral $ABCD$ is cyclic if and only if the circles ω_1 and ω_2 are tangent at their common intersection point Q .

3.2 Part I: The Forward Proof

Assume $ABCD$ as cyclic quadrilateral. It can be proven that O, Q and H are collinear [see Part III].

In quadrilateral $QNOH$,

$$\angle OMQ = \angle ONQ = 90^\circ.$$

As, O is the intersection of perpendicular bisectors of QC, QD and DC . As M is midpoint of QD and QC , this means OM is perpendicular to QD and ON is perpendicular to QC .

This implies,

$$\angle OMQ + \angle ONQ = 90^\circ + 90^\circ = 180^\circ$$

Opposite angles sum to 180° , so $QNOH$ is cyclic quadrilateral.

Also, QO obtains 90° at M , which is on ω_1 .

So, QO is diameter of ω_1 , and α lies on QO , and α is midpoint of QO .

This means, αQ is equal to radius of ω_1 and,

$$\alpha Q = \alpha O = \frac{OQ}{2}$$

As, radius is half of diameter.

In quadrilateral $QFHG$

As, H is the intersection of perpendiculars from vertices A, B, Q onto BQ, CQ , and AC respectively.

This implies,

$$\angle QGH + \angle QFH = 90^\circ + 90^\circ = 180^\circ$$

Opposite angles sum to 180° . So $QFHG$ is cyclic quadrilateral.

Also, QH obtains 90° at G , which is on ω_2 .

So, QH is diameter of ω_2 , and β lies on QH and β is midpoint of QH .

This means, βQ is equal to radius of ω_2 and,

$$\beta Q = \beta H = \frac{HQ}{2}$$

As, radius is half of diameter.

We know that, O, Q and H are collinear, thus these three points lie on same line.

And, α is on line QO , and β is on QH .

This means, α, β are on line OH , thus collinear. —(i)

CASE:1:— If Q lies in between O, H

From relation (i), we know that,

$$\alpha\beta = \alpha Q + \beta Q$$

$$= \text{radius of } \omega_1 + \text{radius of } \omega_2$$

Distance between centers of two circles only equals to sum of their radii if they are externally tangent.

And, as for both diameters of ω_1 and ω_2 , Q is common touching point.

Thus, ω_1 and ω_2 are mutually externally tangent at point Q .

CASE:2:— If Q doesn't lie in between O, H

From relation (i) we can say that,

$$\begin{aligned}\alpha\beta &= |\alpha Q - \beta Q| \\ &= \text{radius of } \omega_1 - \text{radius of } \omega_2\end{aligned}$$

Distance between centers of two circles only equals to difference of their radii if circles are internally tangent.

And, as for both diameters of ω_1 and ω_2 , Q is common touching point.

Thus, ω_1 and ω_2 are mutually internally tangent at point Q .

So we can conclude that ω_1 and ω_2 are mutually tangent $\square$

3.3 Part II: The Converse Proof

From Figure 1, ω_1 and ω_2 are tangent at point Q .

This implies, α, β and Q are collinear and,

$$\begin{aligned}\alpha\beta &= \alpha Q + \beta Q \\ &= \text{sum of radii of two circles. (i)}\end{aligned}$$

if, two circles are externally tangent.

$$\begin{aligned}\text{or, } \alpha\beta &= |\alpha Q - \beta Q| \\ &= \text{difference of radii of two circles. (ii)} \\ &\text{if, two circles are internally tangent.}\end{aligned}$$

QO is diameter of ω_1 .

This means, Q, α and O are collinear. (iii)

Also, QH is diameter of ω_2 .

This means, Q, β and H are collinear. (iv)

From relation (i), (ii), (iii) and (iv) we can say that,

Q, O and H are collinear.

This statement is only true if and only if quadrilateral $ABCD$ is cyclic.

Thus, $ABCD$ is cyclic quadrilateral. $\square$

3.4 Part III: The Collinearity Proof for Q, O and H

From Figure 1, In triangle AQI ,

$$\angle AIQ = 90^\circ$$

As, QI is perpendicular to AB .

This means, angle between QA and AI and angle between AQ and QI sum

$$\angle QAI + \angle IQA = 90^\circ$$

$$\text{or, } \angle IQA = 90^\circ - \angle QAI$$

In ω ,

$$OQ = OC = \text{circumradius of } \triangle QCD.$$

As, O is the circumcenter of triangle QCD .

This means,

$$\angle OQC = \angle OCQ.$$

As, opposite sides are equal.

In triangle OQC ,

$$\angle OQC + \angle OCQ + \angle QOC = 180^\circ$$

$$\text{or, } \angle QOC = 180^\circ - \angle OQC - \angle OCQ$$

$$\text{or, } \angle QOC = 180^\circ - 2\angle OQC$$

As, stated previously, $\angle OQC = \angle OCQ$

$$\text{or, } \frac{1}{2}\angle QOC = 90^\circ - \angle OQC \text{ (i)}$$

Also, in ω ,

QC is a chord and D is a point on circumference of ω on the side of center O

From, Central Angle Theorem,

$$\angle QOC = 2 \times \angle QDC$$

$$\text{or, } \frac{1}{2}\angle QOC = \angle QDC \text{ (ii)}$$

From relation (i) and (ii),

$$\frac{1}{2}\angle QOC = 90^\circ - \angle OQC$$

$$\text{or, } \angle QDC = 90^\circ - \angle OQC$$

$$\text{or, } \angle OQC = 90^\circ - \angle QDC \text{ (iii)}$$

Also, in circumcircle of $ABCD$, BC is a chord, A and D are two distinct points on the same side of chord BC .

From, Angles in the Same Segment Theorem, we can say that,

$$\angle BAC = \angle BDC$$

$$\text{or, } \angle QAI = \angle QDC$$

As, Q is a point on AC and BD , I is a point on AB .

This implies,

$$\angle IQA = 90^\circ - \angle QAI$$

$$\text{or, } \angle IQA = 90^\circ - \angle QDC$$

$$\text{As said earlier, } \angle QAI = \angle QDC.$$

$$\angle IQA = \angle OQC \text{ From relation (iii).}$$

But they are vertically opposite angles with respect to line AC and segment

H is on IQ . Hence, I, H and Q are collinear

From this we can say that, HO is a straight line passing through Q .

As, for two intersecting lines, vertically opposite angles are same.

Thus, H, Q and O are collinear. $\square$

4. Conclusion

In this paper, we introduced a novel characterization of cyclic quadrilaterals their cyclicity is entirely governed by the internal mutual tangency of the two circles ω_1 and ω_2 , at a point Q . This elegant result reveals a deeper underlying local triangle centers (orthocenter and circumcenter) configurations with the quadrilaterals, offering a new perspective for future research into higher-order

5. References

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