

A proof that $\ln(x)$ can not be expressed by the quotient of two polynomials

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This post is a recording of a proof that the logarithm function $\ln(x)$ can not be expressed as the ratio of two polynomials. Actually this can be easily inducted by judging the domain of the function.

Proof: Suppose that $\ln(x)$ can be expressed as $\frac{f(x)}{g(x)}$. We may assume that $f(x) = a_m x^m + \dots + a_0$ and $g(x) = b_n x^n + \dots + b_0$. As we have already known the rule of $\ln(x)$ that $\ln(x^k) = k \ln(x)$, we have

$$\frac{f(x^k)}{g(x^k)} = k \frac{f(x)}{g(x)}$$

holds for every $k \in \mathbb{N}$, that is

$$f(x^k)g(x) - kf(x)g(x^k) = 0, \forall k \in \mathbb{N}.$$

Now consider the first term of polynomial $f(x^k)g(x) - kf(x)g(x^k)$, that is either $a_m b_n x^{km+n}$ or $-ka_m b_n x^{m+kn}$ or $a_m b_n x^{km+n} - ka_m b_n x^{m+kn}$ (only when $k = 1$ or $m = n$). Then we get a contradiction by $a_m b_n \neq 0$.