vulpeculiar musing 14: unique factorization as an injection from multisets

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We can construct a bijection between terms of a magma over A and the set of (finitary) binary trees with leaves labeled in A in an obvious way. Simply define xy as the tree with two leaves labeled x and y, and construct everything inductively. If our magma is commutative, we can rotate each branch around without changing the value of the corresponding term, so it makes sense to consider the corresponding equivalence relation of “mobile trees”, and if our magma is a semigroup we can ignore the branching structure entirely and simply consider the equivalence relation under “forgetting the tree” and regarding only the order structure, which is to say, we obtain sequences. If we combine these impositions, the equivalence classes come to represent multisets.

Then, ignoring units and the identity element which complicate the matter, we can regard an Abelian semigroup A with unique factorization as one with the existence of an injection from multisets over A to A itself in such a way that it respects the semigroup operation via the identification established above.

Actually, the addition of idempotence to the Abelian semigroup (which obtains a semilattce) makes our equivalence classes now represent *sets*. This is a sense in which terms of a semigroup, Abelian semigroup, and semilattce are particularly nice, as they are in bijection with, respectively, sequences, multisets, and sets.

The complication of conventional factorization in its necessary exclusion of units suddenly seems a bit arbitrary from this abstract viewpoint, as it now seems like one is “arbitrarily” including multiplication by a whole family of factors in the equivalence classes, and nontrivial ring-theoretic perspectives are required to “justify” this imposition.

In reality, universal factorization is usually considered only in contexts that make the set of units already well-behaved in some regard.

In any case, while quasigroups only allow universal factorization to be present in trivial cases, we can still consider a tree-derived representation of the terms of structures such as rigs. Even in the extraordinary generality of arbitrary universal algebra, terms can be represented as arbitrary (finitary) trees with non-leaf vertices of degree n each having their own label set. Actually, in the specific case of an algebra with a single ternary operation it might be enlightening to consider, instead of ternary trees with labeled leaves, *binary* trees with *all* nodes labeled.

These representations open up the ability to consider algebra with universal factorization analogues in many different signatures, and without requiring commutativity or associativity, although note that all of the notions discussed heretofore have a cardinality restriction. That is, even in the most restrictive case of a set bijection is only possible in infinite sets, and even then, only due to the restriction to finitary subsets.

In the next musing, we will consider another related association of terms to graphs.