vulpeculiar musing 9: a generalization of lattices with different absorptions

vulpecule • 18 Aug 2026

By analogy with rings and rngs, we will assume all lattices are bounded, and refer to the unbounded case as a “lattce” (pronounced ‘lats’). Explicitly, a lattce is a pair of Abelian semigroups $(\vee, \wedge)$ connected by the absorption axiom $x \vee (x \wedge y) = x \wedge (x \vee y) = x$. It is clear that this absorption forces the component structures to be idempotent and hence semilattices, or rather semilattces.

Now consider the operation defined by $xy = x \vee (x \wedge y)$. Obviously this is simply the left zero band, and this leads to a generalization of latt(i)ces.

A **circumlatt(i)ce** is a pair of semilatt(i)ces wherein the term-defined absorption magma is a band - and since idempotence is inherited from idempotence of join and meet, we actually only need to enforce associativity. That means we could also give it as a pair of semilatt(i)ces connected by the compatibilities $x \vee (x \wedge y) = x \wedge (x \vee y)$ and $(x \vee (x \wedge y)) \vee ((x \vee (x \wedge y) \wedge z) = x \vee (x \wedge (y \vee (y \wedge z)))$. Then a circumlatt(i)ce is a latt(i)ce precisely when the absorption magma is the left-zero band. Note that any semilatt(i)ce forms a circumlatt(i)ce when it is both join and meet, so let us exclude this case as trivial.

There are several interesting cases to narrow in on. A **normal circumlatt(i)ce** is a circumlatt(i)ce whose absorptive band is left normal ($axy = ayx$). A **rectangular circumlatt(i)ce** is a circumlatt(i)ce whose absorptive band is rectangular ($xyx = x$). Every latt(i)ce is both a normal and rectangular circumlatt(i)ce and in fact latt(i)ces are precisely those circumlatt(i)ces which are both normal and rectangular.

You may be tempted to define a **triple circumlatt(i)ce** as a pair of (non-identical) semilatt(i)ces with a commutative absorption band (hence also a semilattce), but in fact all such structures necessarily have identical join and meet.