vulpeculiar musing 4: explorations from quasigroup generalizations

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https://functor.network/user/3486/entry/1921

This musing requires the prerequisites of musing 1, particularly regarding quasigroups.

Let's jump right into defining some generalizations of quasigroups.

First, define the multi-image of a function $MIm(f)$ as the multiset over $Im(f)$ where the multiplicity of x is the number of solutions y to $f(y) = x$.

In a quasigroup, all of these multiplicities are by definition 1 (and in particular the image is the entire quasigroup), but in our first generalization we simply require the identity $MIm(\ell_a) = MIm(r_b)$ for all a, b. I am sure this generalization has been considered many times before, but I know of no established name for it, so we will call it a **permutation square**, because it captures the intuitive idea of a structure whose rows and columns are permutations of each other.

A similar but weaker condition is that the induced multisets are *isomorphic*, which is to say that the images do not necessarily coincide, but the multiplicities are in one-to-one correspondence. Explicitly, we need the existence, for some arbitrarily chosen a: a bijection ϕ_b, for any b, such that the multiplicity of x in $MIm(\ell_a)$ is the same as the multiplicity of $\phi_b(x)$ in $MIm(r_b)$; and a bijection ψ_b, for any b, such that the multiplicity of x in $MIm(\ell_a)$ is the same as the multiplicity of $\psi_b(x)$ in $MIm(\ell_b)$. That's rather complicated, but the intuitive version is easy enough to understand: the rows and columns have the same frequency distributions of elements. Another way to put it: if there are k distinct elements that occur n times in row a, there will also be k distinct elements that occur n times in column b. This we will call a **frequency square**, although I believe the extant usage of the term is slightly different.

I am pretty sure that we can axiomatize frequency squares so that they form a quasivariety:

- $xy = xz$ implies $w(w \circ_\ell y) = w(w \circ_\ell z)$
- $xy = xz$ implies $(x \circ_r y)x = (x \circ_r z)x$
- The above can be replaced with $yx = zx$ implying $(w \circ_r y)w = (w \circ_r z)w$ if we don't require the left and right multisets to coincide.
- $x \bullet_\ell (x \circ_\ell y) = y$
- $x \bullet_r (x \circ_r y) = y$.

There are three ways to strengthen a frequency square without getting to a permutation square (or at least three that we consider).

- First, a **semipermutation square** is a frequency square wherein every one of our multiplicity-preserving isomorphisms is an involution - and since the composition of involutions is not necessarily an involution, this requires that we enforce the existence of bijections for every element and not just a single arbitrarily chosen one. Since the identity is clearly an involution, a permutation square is a semipermutation square.
- Second, a **strong frequency square** is a frequency square that preserves the indices of induced functions, i.e. $\phi_b(a) = b$. This, like the above, requires explicitly imposed bijections for every element, and then the property can be expressed as an identity $\phi_{a \to b}(a) = b$. More simply stated, this means that a has the same multiplicity in row/column a as b does in row/column b.
- Third, a **coherent frequency square** is a frequency square where the same pre-images remain in the same frequency class, i.e., if $\ell_a(x) = \ell_a(y)$, then $\ell_b(x) = \ell_b(y)$, for all b. This condition has also surely been considered before, and in fact I believe it is actually equivalent to the condition of having no nontrivial proper left- or right-ideals.

These notions inspire a strengthening of frequency square that is actually incomparable with quasigroups: that of a **function square**, where any pair of induced functions are isomorphic. To wit, we need the existence, for some arbitrarily chosen a: a bijection ϕ_b, for any b, such that $\phi_b(\ell_a(x)) = r_b(x)$; and a bijection ψ_b, for any b, such that

$\psi_b(\ell_a(x)) = \ell_b(x)$. Note that the concepts of strength and coherence apply equally well, and there is even an analogue of semipermutation square available, which we shall term **semiconstant magma**.

In any case, a quasigroup is clearly a permutation square, and while a permutation square is not necessarily strong or coherent, a quasigroup is in a trivial way, since every element has multiplicity one and therefore any bijection will suffice for its multiset isomorphisms, and then bijections with the sought properties may be chosen easily.

There is a lot to explore here.