vulpeculiar musing 4: explorations from quasigroup generalizations

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First, define the functions induced by left multiplication by a via $\ell_a(x) = xa$, and the functions induced by right multiplication by a via $r_a(x) = ax$. The condition of being a row-quasigroup is then that ℓ_a is a bijection for all a, etc. However, instead we now define the multi-image of a function $MIm(f)$ as the multiset over $Im(f)$ where the multiplicity of x is the number of solutions y to $f(y) = x$.

In a quasigroup, all of these multiplicities are by definition 1 (and in particular the image is the entire quasigroup), but in our generalization we simply require the identity $MIm(\ell_a) = MIm(r_b)$ for all a, b. I am sure this generalization has been considered many times before, but I know of no established name for it, so we will call it a **permutation square**.

A similar but weaker condition is that the induced multisets are *isomorphic*, which is to say that the images may not coincide, but the multiplicities are in one-to-one correspondence. Explicitly, we need the existence, for some arbitrarily chosen a: a bijection ϕ_b, for any b, such that the multiplicity of x in $MIm(\ell_a)$ is the same as the multiplicity of $\phi_b(x)$ in $MIm(r_b)$; and a bijection ψ_b, for any b, such that the multiplicity of x in $MIm(\ell_a)$ is the same as the multiplicity of $\psi_b(x)$ in $MIm(\ell_b)$. That's rather complicated, but the intuitive version is easy enough to understand: the rows and columns have the same frequency distributions of elements. This we will call a **frequency square**, although I believe the extant usage of the term is slightly different.

I am pretty sure that we can axiomatize frequency squares so that they form a quasivariety:

- $xy = xz$ implies $w(w \circ_\ell y) = w(w \circ_\ell z)$
- $xy = xz$ implies $(x \circ_r y)x = (x \circ_r z)x$
- The above can be replaced with $yx = zx$ implying $(w \circ_r y)w = (w \circ_r z)w$ if we don't require the left and right multisets to coincide.

- $x \bullet_\ell (x \circ_\ell y) = y$
- $x \bullet_r (x \circ_r y) = y$.

There are three ways to strengthen a frequency square without getting to a permutation square (or at least three that we consider).

- First, a **semipermutation square** is a frequency square wherein every isomorphism is an involution - and since the composition of involutions is not necessarily an involution, this requires that we enforce the existence of bijections for every element and not just a single arbitrarily chosen one. Since the identity is clearly an involution, a permutation square is a semipermutation square.
- Second, a **strong frequency square** is a frequency square that preserves the indices of induced functions, i.e. $\phi_b(a) = b$. This, like the above, requires explicitly imposed bijections for every element, and then the property can be expressed as an identity $\phi_{a \to b}(a) = b$.
- Third, a **coherent frequency square** is a frequency square where the same pre-images remain in the same frequency class, i.e., if $\ell_a(x) = \ell_a(y)$, then $\ell_b(x) = \ell_b(y)$, for all b. This condition has also surely been considered before, and in fact I believe it is actually equivalent to the intersection of left and right simplicity in semigroups?

These notions inspire a strengthening of frequency square that is actually incomparable with quasigroups: that of a **function square**, where any pair of induced functions are isomorphic. To wit, we need the existence, for some arbitrarily chosen a: a bijection ϕ_b, for any b, such that $\phi_b(\ell_a(x)) = r_b(x)$; and a bijection ψ_b, for any b, such that $\psi_b(\ell_a(x)) = \ell_b(x)$. Note that the concepts of strength and coherence apply equally well, and there is even an analogue of semipermutation square available, which we shall term **semiconstant magma**.

In any case, a quasigroup is clearly a permutation square, and while a permutation square is not necessarily strong or coherent, a quasigroup is in a trivial way, since every element has multiplicity one, any bijection will suffice for its multiset isomorphisms.

There is a lot to explore here.