vulpeculiar musing 3: smallest counterexample algebra in the variety of bands

vulpecule · 10 Aug 2026

https://functor.network/user/3486/entry/1920

This musing requires the prerequisites of musing 1, and the knowledge that a band is an idempotent (that is, $x^2 = x$) semigroup (that is, it is associative as well).

In the literature on subvarieties of the variety of bands, there is a series of identities based on equivalences between certain recursively-defined words.

In particular, the G_n-words are defined as $x_n G^*_{n-1}$, with * indicating the dual - that is, reversal - and the base case $G_1 = x_1$, and the I_n-words are defined as $G_n x_n I^*_{n-1}$, with the base case $I_2 = x_2 x_1 x_2$. Let's call bands satisfying $G_n = I_n$ as GI_n-bands.

In particular, consider GI_5-bands. Let's build this up one-by-one:

- $G_2 = x_2 x_1$
- $G_3 = x_3 x_1 x_2$
- $G_4 = x_4 x_2 x_1 x_3$
- $G_5 = x_5 x_3 x_1 x_2 x_4$
- $I_3 = x_3 x_1 x_2 x_3 x_2 x_1 x_2$
- $I_4 = x_4 x_2 x_1 x_3 x_4 x_2 x_1 x_2 x_3 x_2 x_1 x_3$
- $I_5 = x_5 x_3 x_1 x_2 x_4 x_5 x_3 x_1 x_2 x_3 x_2 x_1 x_2 x_4 x_3 x_2 x_4$.

Using Mace4, we can observe that the (unique) smallest band which is not a GI_2-band is of order two, the (unique) smallest band which is not a GI_3-band is of order 5, and the (unique) smallest band which is not a GI_4-band is of order 9. What is the order of the smallest band which is not a GI_5-band, and is it unique?

Also, are these smallest counterexamples possibly free algebra? The concept of a free algebra is recalled in musing 2.