vulpeculiar musing 2: how nice are varieties with finite free models?

vulpecule · 10 Aug 2026

https://functor.network/user/3486/entry/1918

This musing contains content which may be extremely basic universal algebra knowledge, immediately to be cleared up once I learn more UA. Also, the background on magmata and varieties from the previous musing is used.

To understand the concept of a free algebra, first we need to consider the set of all expressions in some algebraic language (that is, signature), over some alphabet. For example, in the signature of magmata (one binary operation) and the alphabet with one letter (say, a), the set of all expressions is just the set of all bracketings of a finite string of repeated a s. Now, note that this structure itself can be easily equipped with a magma, by simply taking the product of expressions $P \cdot Q$ to simply be the expression consisting of the term P multiplied by the term Q, and the same can be said for any other signature. This structure is the free algebra of the variety given by that signature with no equational axioms assumed - let's simply take this to be definitional. The free algebra of any variety (in a given signature, over some alphabet) can then be regarded as the structure that results from taking that initial structure and making the identifications of expressions suggested by the identities the variety satisfies. For instance, the free semigroup over an alphabet is the set of all finite words in that alphabet because the various bracketings are all identified via the defining equation $x \cdot yz = xy \cdot z$.

Now, it is of particular interest when a variety has a finite free algebra. Some varieties' free algebra are only finite up to a certain alphabet size (consider for instance the free algebra of an idempotent magma), while some others, termed *locally finite* varieties, have free algebra for all finite alphabet sizes. Is it true that any subvariety of a locally finite variety is locally finite? This certainly seems like it should be true. In fact, it seems like something much stronger should be true - if the free algebra of variety $\mathcal{V}$ on an alphabet of size n has k elements (clearly $n \leq k$), and $\mathcal{W}$ is a subvariety of $\mathcal{V}$, then the free algebra of $\mathcal{W}$ on an alphabet of size n has $m \leq k$ elements. Does my intuition hold up?