vulpeculiar musing 1: nonidempotent Latin quandles

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This musing assumes the reader is familiar with the concepts of variety of algebra, magma, and quasigroup, and in particular the Birkhoff axiomatization of quasigroup that allows them to form a variety. Additionally, nonassociative algebraic conventions of operator precedence will also be utilized, and whenever I speak of enumerations, uniquenesses, or distinctions among magmata, it will be up to isomorphism.

As will be usual for me, I will dispense with much of the motivation for concepts discussed, as by and large I don't care about such things. This math is perhaps "too pure" - be warned.

A **shelf** is a magma which satisfies an "auto-distributivity" property on either the left - $x \cdot yz = xy \cdot xz$, or the right - $yz \cdot x = yx \cdot zx$. A **rack** is a shelf which is either a left or right quasigroup, matching the directionality of the auto-distributivity - that is, a left rack is a row-quasigroup and a right rack is a column-quasigroup. If a rack is both a row- and column-quasigroup, it is said to be a **Latin rack**. A **quandle** is a rack which is idempotent - $x^2 = x$. If it is in fact a Latin rack, it is naturally said to be a **Latin quandle**.

Now, it may not be immediately apparent, but it turns out that all Latin racks are Latin quandles. To show this, consider the auto-distributivity equation (let's arbitrarily choose the left), and say $z = y = x$, so we have $x \cdot x^2 = x^2 \cdot x^2$. But this is of the form $a \cdot z = b \cdot z$, thus column-cancellativity allows us to conclude that $a = b$ and $x = x^2$. This holds just as well on the right - thus every Latin rack is a quandle.

This brings up a question: can we come up with a variety of non-idempotent algebras, for which the variety of Latin quandles form precisely the proper subvariety characterized by idempotence? More simply, can we have some basis of equations that neither includes nor implies idempotence, such that adding idempotence to them defines precisely Latin quandles? Yes, in fact it is quite easy: for example, one may simply take the auto-distributivity equation and replace some of its terms with squares. This clearly would recover auto-distributivity in the idempotent case, so all that needs to be checked is that there are non-idempotent examples.

Let's try, again utilizing left autodistributivity, and just arbitrarily replace some term with a square, and then considering non-idempotent quasigroups satisfying the new identity:

- $x^2 \cdot yz = xy \cdot xz$
- $x \cdot yz^2 = xy \cdot xz$
- $x \cdot (yz)^2 = xy \cdot xz$
- $x \cdot yz = (xy)^2 \cdot xz.$

With the help of Mace4, we find that…

- A two-element non-idempotent quasigroup satisfying the first equation is actually given by Boolean XNOR, and we can find several examples of higher order as well. This equation in particular is interesting as it is a *regular* equation.
- No non-idempotent two- or three-element quasigroups satisfying the second equation exist, but we have a unique four-element example, alongside unique examples of orders five and nine at least:
$$\begin{array}{cccc} 1 & 2 & 0 & 3 \\ 3 & 0 & 2 & 1 \\ 2 & 1 & 3 & 0 \\ 0 & 3 & 1 & 2 \end{array}$$
.
- No two-, four-, six-, or eight-element non-idempotent quasigroups satisfying the third equation exist, but we have this given (unique) three-element example, along with (unique) five-, (unique) seven-, and (non-unique) nine-element examples:
$$\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{array}$$
.
- No two-, three-, five-, six-, seven-, or nine-element non-idempotent quasigroups satisfying the fourth equation exist, but the four-element example above also (uniquely) satisfies this one, and there are two examples of order eight.

Some questions raised:

- What orders actually possess non-idempotent quasigroups satisfying the second equation, and do any have more than one example?

- What orders actually possess non-idempotent quasigroups satisfying the third equation? Do all even orders have examples? Do any odd orders have examples?
- What orders actually possess non-idempotent quasigroups satisfying the fourth equation? Do all orders divisible by 4 have examples? Do any orders not divisible by 4 have examples?
- Every example of the latter three equations I could find has unique square roots. Does this always hold?
- There were six examples of order nine of the third equation. Are these parastrophically equivalent?