vulpeculiar musing 1: nonidempotent Latin quandles

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Note that whenever I speak of enumerations, uniquenesses, or distinctions among magmata, it will be up to isomorphism.

As will be usual for me, I will dispense with much of the motivation for concepts discussed, as by and large I don't care about such things. This math is perhaps "too pure" - be warned.

A **magma** on A is simply a function $f : A^2 \to A$, which is typically regarded as an "operation" on A, written with infixes, like addition $(x + y)$ or multiplication $a \times b$. In particular, though, in the absence of any other structure or interpretation of the symbol, it is written (semi-)interchangeably as $x \cdot y$ and simple juxtaposition xy, and refer to it as if it were multiplication (that is, using words such as 'product' or 'factors'). Explicitly, we define $x \cdot y = f(x, y)$. It is important to remember that despite the multiplicative convention, the magmatic operation is not necessarily commutative nor associative. In fact, we subscribe to a common convention from nonassociative algebra that juxtaposition binds more tightly than $\cdot$ but refers to the same operation, so that instead of $(xy)z = x(yz)$ we write $xy \cdot z = x \cdot yz$. We may also freely use the exponential notation for squares, i.e., $x^2 = x \cdot x$, and this binds even tighter than juxtaposition. However, since $xx^2 = x^2x$ is not true in general, cubes and above are not defined and we will only use them when that equation (cube-associativity) *does* hold. In the literature, one does find the convention that powers associate on the left (or more confusing notions, so presumably the simple opposite of associating on the right is present as well) but we will not use this or any other, and simply not use such expressions other than the unambiguous square.

Notice that we may also view a magma as being defined by a certain kind of matrix or "multiplication table", where the product xy is located in the cell that is the intersection of the row labeled x and the column labeled y.

(Well, this view is somewhat problematic in the infinite and especially uncountable case, but there is no issue among finite examples and this is usually what is discussed.)

A **quasigroup**, then, is simply the same thing as a Latin square, appropriately abstracted. That is, in every row every element appears precisely once (*left* quasigroup), and in every column every element appears precisely once (*right* quasigroup). This is the simple intuition for the concept, but formally it is not very nice. There are two primary definitions for a quasigroup, and it is important to understand both of them. Firstly, a quasigroup can be regarded as a magma which is both *cancellative* and *divisible*, on both sides. A magma is left-cancellative if $xy = xz$ implies $y = z$, or intuitively "an element appears at most once in a row", and left-divisible if, for any given pair x, y, there exists some z satisfying $xz = y$, or intuitively "every element appears in every row". These definitions recover the above notion on the left, and it is easy to see how to apply it to the right side as well.

However, there is a reason that this definition might be displeasing in certain contexts. Namely, it contains an implicative statement, and (even worse) an existential quantification. It is often more useful to consider only algebraic structures whose definitions contain only universally quantified equations, and classes of such structures are called *varieties of algebra*, or usually simply varieties. (Note that this should not be confused with the completely unrelated concept of an algebraic variety. The two terms are not used together, but the existence of both can be confusing.) Many algebraic structures defined with implications or existential quantifiers turn out not to be varieties, i.e., there is no list of equations that precisely characterizes them, but this is not actually the case for quasigroups.

That is, there is in fact a quite simple set of equations that defines the quasigroups, but it requires one to explicitly define other operations than just the magmatic "multiplication" that are derivable from the previous definition. Some thought might allow you to see that you need to explicitly define "division" equationally: we need to have $xy \div_r y = x$ and $xy \div_\ell x = y$. In any case, though, all you need to remember is that quasigroups *do* form a variety.

Now, moving on - a **shelf** is a magma which satisfies an "auto-distributivity" property on either the left - $x \cdot yz = xy \cdot xz$, or the right - $yz \cdot x = yx \cdot zx$. A **rack** is a shelf which is either a left or right quasigroup, matching the directionality of the auto-distributivity - that is, a left rack is a row-quasigroup and a right rack is a column-quasigroup. If a rack is both a row- and column-quasigroup, it is said to be a **Latin rack**. A **quandle** is a rack which is idempotent - $x^2 = x$. If it is in fact a Latin rack, it is naturally said to be a **Latin quandle**.

Now, it may not be immediately apparent, but it turns out that all Latin racks are Latin quandles. To show this, consider the auto-distributivity equation (let's arbitrarily choose the left), and say $z = y = x$, so we have $x \cdot x^2 = x^2 \cdot x^2$. But this is of the form $a \cdot z = b \cdot z$, thus column-cancellativity allows us to conclude that $a = b$ and $x = x^2$. This holds just as well on the right - thus every Latin rack is a quandle.

Since quasigroups form a variety, and auto-distributivity and idempotence are equational properties, it follows that Latin quandles also form a variety, but the above means that idempotence is not a necessary axiom for the equational presentation of that variety. This brings up a question: can we come up with a variety of non-idempotent algebras, for which the variety of Latin quandles form precisely the proper subvariety characterized by idempotence? More simply, can we have some basis of equations that neither includes nor implies idempotence, such that adding idempotence to them defines precisely Latin quandles?

Yes, in fact it is quite easy: for example, one may simply take the auto-distributivity equation and replace some of its terms with squares. This clearly would recover auto-distributivity in the idempotent case, so all that needs to be checked is that there are non-idempotent examples.

Let's try, again utilizing left autodistributivity, and just arbitrarily replace some term with a square, and then considering non-idempotent quasigroups satisfying the new identity:

- $x^2 \cdot yz = xy \cdot xz$
- $x \cdot yz^2 = xy \cdot xz$
- $x \cdot (yz)^2 = xy \cdot xz$
- $x \cdot yz = (xy)^2 \cdot xz$.

With the help of Mace4, we find that…

- A two-element non-idempotent quasigroup satisfying the first equation is actually given by Boolean XNOR, and we can find several examples of higher order as well. This equation in particular is interesting as it is a *regular* equation.
- No non-idempotent two- or three-element quasigroups satisfying the second equation exist, but we have a unique four-element example, alongside unique examples of orders five and nine at least:

$$\begin{matrix} a & b & c & d \\ d & c & b & a \\ b & a & d & c \\ c & d & a & b \end{matrix} .$$

- No two-, four-, six-, or eight-element non-idempotent quasigroups satisfying the third equation exist, but we have this given (unique) three-element example, along with (unique) five-, (unique) seven-, and (non-unique) nine-element examples:

$$\begin{matrix} a & b & c \\ b & c & a \\ c & a & b \end{matrix} .$$

- No two-, three-, five-, six-, seven-, or nine-element non-idempotent quasigroups satisfying the fourth equation exist, but the four-element example above also (uniquely) satisfies this one, and there are two examples of order eight.

Some questions raised:

- What orders actually possess non-idempotent quasigroups satisfying the second equation, and do any have more than one example?
- What orders actually possess non-idempotent quasigroups satisfying the third equation? Do all even orders have examples? Do any odd orders have examples?
- What orders actually possess non-idempotent quasigroups satisfying the fourth equation? Do all orders divisible by 4 have examples? Do any orders not divisible by 4 have examples?
- Every example of the latter three equations I could find has unique square roots. Does this always hold?

- There were six examples of order nine of the third equation. Are these parastrophically equivalent?