Homotopy Groups of Spheres via Morse Theory and Differential Topology

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Calculating higher homotopy groups of topological spaces is notoriously difficult. Even for something as simple as spheres—the fundamental building blocks of shape—we lack a complete formula for all higher homotopy groups.

Yet, amidst this wild complexity, there is a remarkable organizing principle: **they stabilize eventually**. Formally, the Freudenthal Suspension Theorem guarantees that the homotopy groups $\pi_{n+k}(S^n)$ are eventually constant for $n \geq k+2$, allowing us to define the stable homotopy groups of spheres via the direct limit:

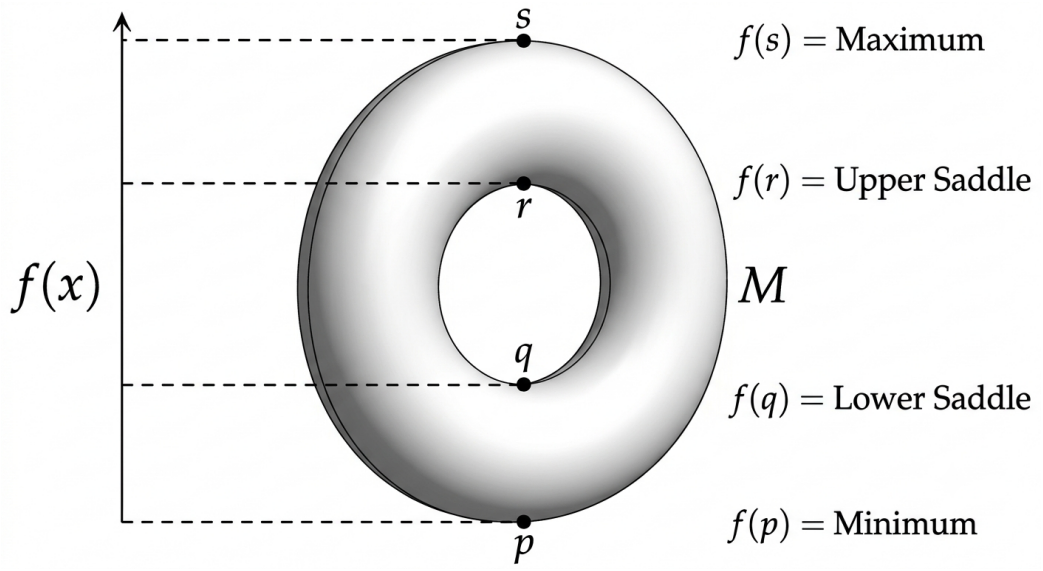
$$\pi_k^s = \varinjlim_{n \to \infty} \pi_{n+k}(S^n)$$

In this blog post, we are going to prove this result, but not through standard algebraic machinery. Instead, we will use methods from Morse theory—drawing on the geometric framework in Milnor's classic text—to build our proof from the ground up, introducing the necessary Morse-theoretic basics along the way.

In the second half of the post, we will shift to a completely different vantage point: higher homotopy groups through the lens of the **Pontryagin-Thom construction**. This geometric perspective translates abstract homotopy classes into framed cobordisms, giving us a deeply visual way to understand the stable homotopy groups of spheres.

Introduction to Finite Dimensional Morse Theory

To understand how Morse theory provides a powerful toolkit for topology, let us begin with a classic, intuitive example: the standard torus $M \subset \mathbb{R}^3$ standing upright, equipped with the height function $f : M \to \mathbb{R}$ given by projection onto the vertical axis.



For any real number a, let us define the sublevel set:

$$M^a = \{x \in M \mid f(x) \leq a\}$$

As we sweep the level parameter a from the bottom of the torus to the top, the topology of the sublevel sets M^a evolves. There are four critical points of the height function h where the derivative vanishes, corresponding to the bottom-most minimum, the two saddle points on the sides, and the top-most maximum. Let us label these critical points:

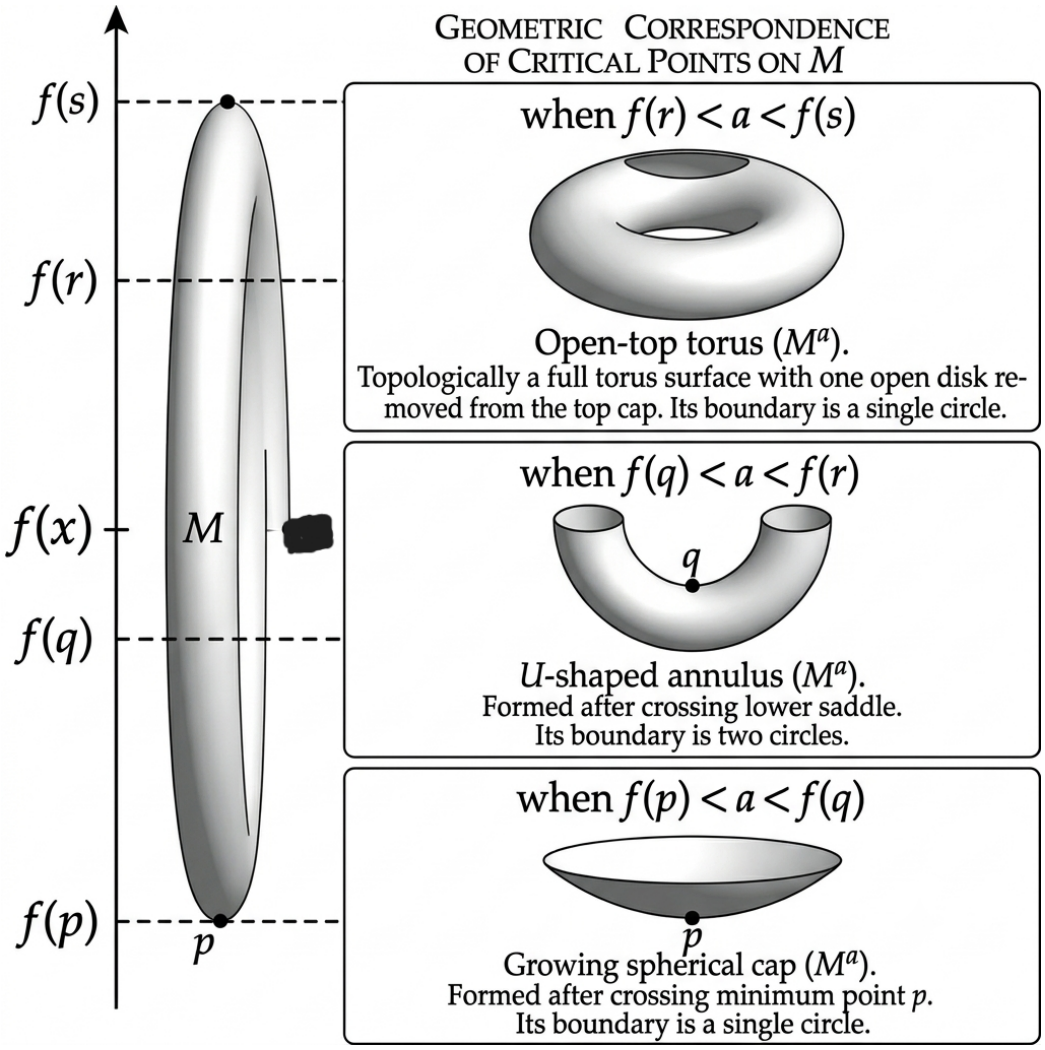
- p: The local minimum at the very bottom ($f(p) = \min$), where M^a is initially empty and then becomes a small disk.
- q: The lower saddle point on the inner equator.
- r: The upper saddle point on the outer equator.
- s: The local maximum at the very top.

The Evolution of Sublevel Sets (M^a)

Let's examine how the topology of M^a changes as a varies across different intervals:

1. Between Critical Values (Regular Intervals)

- When a moves through values where no critical point is crossed (i.e., between the critical values $f(p)$, $f(q)$, $f(r)$, and $f(s)$), the topology of M^a does not change at all. Specifically, if there are no critical points in the interval $[a_1, a_2]$, the sublevel sets M^{a_1} and M^{a_2} are diffeomorphic.



2. Crossing a Critical Value (The Topological Transition)

- It is exceptionally interesting to see what happens precisely when a crosses one of the critical values $f(p), f(q), f(r), f(s)$. Away from critical points, the diffeomorphism type is preserved, but at a critical point, the homotopy type of M^a undergoes a dramatic structural change.
- **Case (i) — Crossing the lower saddle q ($a = f(q)$):** As a increases past $h(q)$, the sublevel set changes its homotopy type by attaching a 1-cell (handle). Topologically, the neck of the torus pinches and opens up, changing a cylinder into a pair of pants or connecting components.
- **Case (ii) — Crossing the upper saddle r ($a = f(r)$):** Similarly, as a crosses $h(r)$, the homotopy type changes again by attaching another 1-cell. This handles the transition where the two loops merge back into a single surface before closing off at the maximum s.

Definitions and Theorems in Morse Theory

Building upon our intuitive geometric example of the upright torus, we can now formalize the foundational concepts and theorems of finite-dimensional Morse theory.

Definition: Morse Function and Nondegenerate Critical Points

A smooth function $f : M \to \mathbb{R}$ on a smooth manifold M is called a **Morse function** if all of its critical points (points where the differential $df_p = 0$) are **nondegenerate**.

A critical point $p \in M$ is nondegenerate if the Hessian matrix of second partial derivatives evaluated at p,

$$\text{Hess}_f(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)$$

is non-singular (i.e., its determinant is non-zero, meaning it has no zero eigenvalues). Consequently, nondegenerate critical points are isolated.

Definition: Index of a Critical Point

If p is a nondegenerate critical point of f, the **Morse index** of f at p (often denoted $\lambda(p)$) is defined as:

- The number of independent directions at p in which the function decreases (i.e., the number of negative eigenvalues of the Hessian matrix).
- Equivalently, the dimension of the maximum linear subspace of the tangent space T_pM on which the Hessian quadratic form is negative definite.

For our upright torus height function:

- Global minimum p: Index 0 (zero negative eigenvalues).
- Lower saddle q: Index 1 (one negative eigenvalue).
- Upper saddle r: Index 1 (one negative eigenvalue).
- Global maximum s: Index 2 (two negative eigenvalues).

The Morse Lemma and Fundamental Theorem of Morse Theory

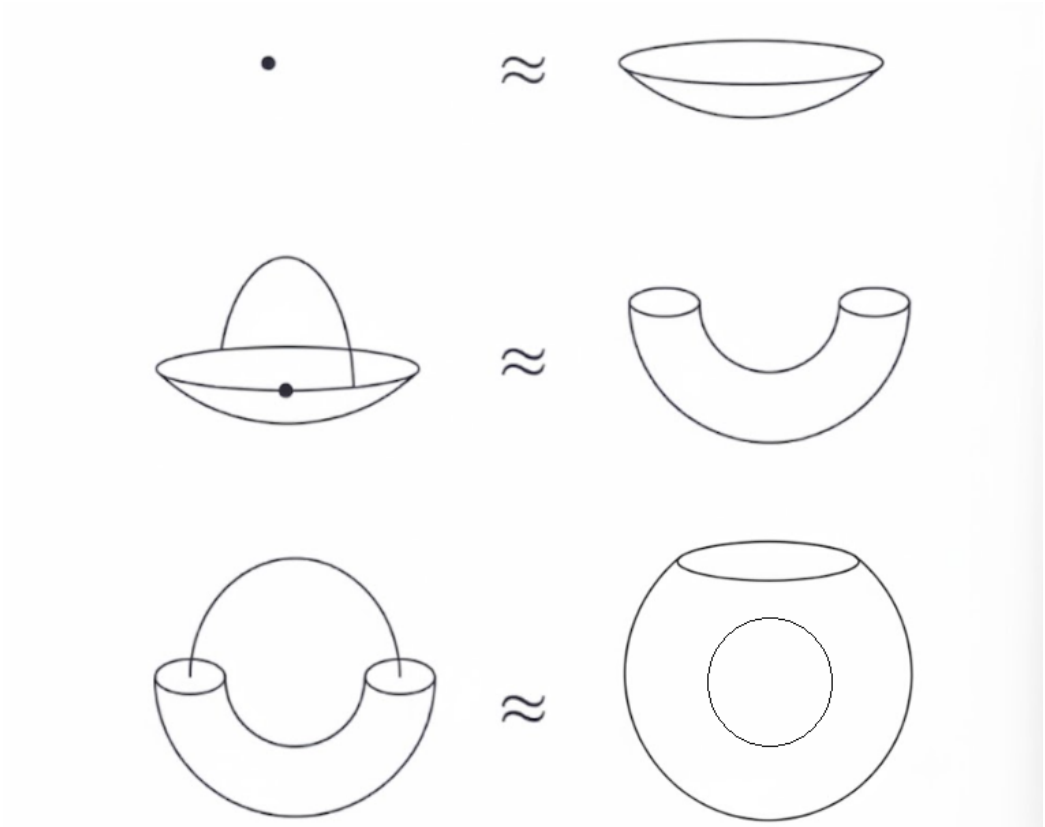
Now, we know from our knowledge of differential topology that at points where we have immersions and submersions, it is essentially an inclusion and projection respectively by some coordinate change. An analogous theorem holds for nondegenerate critical points, which we call the Morse Lemma.

> **Morse's Lemma :** Let p be a non-degenerate critical point for a smooth function f. Then there is a local coordinate system $(y_1, \ldots, y_n)$ in a neighborhood U of p with $y_i(p) = 0$ for all i such that the identity
>
> $$f = f(p) - y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_n^2$$
>
> holds throughout U, where λ is the index of f at p.

Now we state the fundamental theorem of finite dimensional morse theory (there will be an analog for infinite dimensions later), which is really a generalization of what we had already seen in the example of the height function of the torus.

> **Fundamental Theorem:** Let $f : M \to \mathbb{R}$ be a smooth function, and let p be a non-degenerate critical point with index λ. Setting $f(p) = c$, suppose that $f^{-1}[c - \varepsilon, c + \varepsilon]$ is compact, and contains no critical point of f other than p, for some $\varepsilon > 0$. Then, for all sufficiently small ε, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached.

As an illustration of the Fundamental Theorem, the picture below is the cell decomposition for the height function on the torus as it passes the nondegenerate critical points of index 1,1.



adding 1 cell successively for the critical points of index 1,1 on the torus.

Now, having discussed Morse functions, one might intuitively think that such functions are rare or sparse. In reality, the exact opposite is true: we can obtain a plethora of Morse functions by slightly perturbing any smooth function, as guaranteed by the following result:

> **Lemma :** Any bounded smooth function $f : M \to \mathbb{R}$ can be uniformly approximated by a smooth function g which has no degenerate critical points. Furthermore, g can be chosen so that the i-th derivatives of g on the compact set $K \subset M$ uniformly approximate the corresponding derivatives of f, for $i \leq k$.

We can do even better: instead of arbitrary approximations, we can choose g to be a Morse function that is sufficiently close to f, ensuring that the index of critical points is preserved. This stability is guaranteed by the following lemma:

Let K be a compact subset of the Euclidean space $\mathbb{R}^n$; let U be a neighborhood of K; and let

$$f : U \longrightarrow \mathbb{R}$$

be a smooth function such that all critical points of f in K have index $\geq \lambda$.

> **Lemma :** If $g : U \to \mathbb{R}$ is any smooth function which is "close" to f, in the sense that
>
> $$\left|\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\right| < \varepsilon, \quad \left|\frac{\partial^2 g}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_i \partial x_j}\right| < \varepsilon, \quad (i, j = 1, > \ldots, n)$$
>
> uniformly throughout K, for some sufficiently small constant ε, then all critical points of g in K have index $\geq \lambda$.

Note that f is allowed to have degenerate critical points. In the application, g will be a nearby function without degenerate critical points.

Why Morse Theory on Path and Loop Spaces?

So far, we have explored finite-dimensional Morse theory on compact manifolds like our upright torus. But why take this machinery further?

If our ultimate goal is to understand the **higher homotopy groups** $\pi_k(M)$ of a manifold M which capture complex, multi-dimensional topological structures that the fundamental group alone cannot see our objective is to establish the fundamental topological isomorphism relating a manifold to its loop space:

> **Theorem (Loop Space Isomorphism):** For any path-connected manifold M and base point $x_0 \in M$, there is a natural isomorphism of homotopy groups:
>
> $$\pi_{i+1}(M) \cong \pi_i(\Omega M)$$

This identity reveals a profound shift in perspective: instead of tackling the difficult higher homotopy groups of a finite-dimensional manifold M directly, we can study the homotopy groups of its infinite-dimensional loop space ΩM instead.To analyze the topology of ΩM, we need a version of Morse theory that can operate on infinite-dimensional spaces. This exact motivation leads us to adapt Morse theory for path and loop spaces, where geodesics will emerge as the critical points of the energy functional.

Fibrations and Algebraic Topology Prelims

Definition: Hurewicz Fibration

A continuous map $p : E \to B$ is called a **Hurewicz fibration** if it satisfies the homotopy lifting property with respect to any space X: for any homotopy $H : X \times [0,1] \to B$ and any map $\tilde{H}_0 : X \times \{0\} \to E$ such that $p \circ \tilde{H}_0 = H|_{X \times \{0\}}$, there exists a lifted homotopy $\tilde{H} : X \times [0,1] \to E$ such that $p \circ \tilde{H} = H$ and $\tilde{H}|_{X \times \{0\}} = \tilde{H}_0$.

Remark: Serre Fibrations

Serre fibrations are a weaker version of Hurewicz fibrations, requiring the homotopy lifting property to hold only for the class of CW-complexes (or cubes I^k) rather than all topological spaces.

Corollaries on Fibrations

- **Global Products:** Any product projection map $p : F \times B \to B$ given by $p(x, b) = b$ is a Hurewicz fibration.
- **Local-to-Global via Partitions of Unity:** If the base space B is paracompact and a continuous map $p : E \to B$ is a local Hurewicz fibration, then p is a Hurewicz fibration globally using partition of unity techniques.
- **Fiber Bundles:** Since fiber bundles are locally trivial product projections and base spaces of manifolds are paracompact, every locally trivial fiber bundle with a paracompact base is a Hurewicz fibration.

The Long Exact Sequence and the Path-Loop Fibration

Fibrations are uniquely valuable because they give us a powerful long exact sequence of homotopy groups:

> **Theorem (Long Exact Sequence of Homotopy Groups):** Let $p : E \to B$ be a Hurewicz fibration with fiber $F = p^{-1}(b_0)$ over a base point $b_0 \in B$. Then there is a natural long exact sequence:
>
> $$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_0(E) \to \pi_0(B)$$

To prove our target isomorphism, we construct a canonical fibration using the path space and loop space of M. Let PM denote the path space of M (all continuous paths starting at the fixed base point x_0), and let ΩM be the loop space of M (paths that both start and end at x_0).

Consider the evaluation map at the endpoint:

$$p : PM \to M, \quad \gamma \mapsto \gamma(1)$$

- **The Fiber:** The fiber of this map over the base point x_0 is precisely the set of paths that start and end at x_0, which is by definition the loop space ΩM.
- **Contractibility of Total Space:** The path space PM is contractible to a constant path.

Applying the long exact sequence of homotopy groups to the fibration $\Omega M \to PM \to M$:

$$\cdots \to \pi_n(PM) \to \pi_n(M) \to \pi_{n-1}(\Omega M) \to \pi_{n-1}(PM) \to \dots$$

Since PM is contractible, all of its homotopy groups $\pi_n(PM)$ vanish for $n \geq 0$. By exactness of the sequence, the boundary homomorphism $\partial : \pi_n(M) \to \pi_{n-1}(\Omega M)$ becomes an isomorphism for all $n \geq 1$.

Thus, shifting the index yields:

$$\pi_{i+1}(M) \cong \pi_i(\Omega M)$$

This exact sequence and fibration framework validate our infinite-dimensional strategy: analyzing the topology of ΩM through Morse theory directly unlocks the higher homotopy groups of M, where **geodesics** will emerge as critical points of the energy functional!

Morse Theory on Path Spaces

Having established the algebraic topology foundations and the motivation for using infinite-dimensional spaces, we now transition to the framework of Morse theory on path spaces.

Piecewise Smooth Path Spaces

Let M be a smooth manifold and let p and q be two (not necessarily distinct) points of M. By a **piecewise smooth path** from p to q will be meant a map

$\omega : [0,1] \to M$ such that:

- There exists a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ so that each $\omega|_{[t_{i-1},t_i]}$ is differentiable of class C^∞.
- $\omega(0) = p$ and $\omega(1) = q$.

The set of all piecewise smooth paths from p to q in M will be denoted by $\Omega(M; p, q)$, or briefly by $\Omega(M)$ or Ω. Later (in Chapter 16) Ω will be given the structure of a topological space, but for the moment this will not be necessary. We will think of Ω as being something like an ``infinite dimensional manifold''.

Variations of Paths and Vector Fields

To start the analogy, by the **tangent space of Ω at a path** ω will be meant the vector space consisting of all piecewise smooth vector fields W along ω for which $W(0) = 0$ and $W(1) = 0$. The notation $T\Omega_\omega$ will be used for this vector space.

> **Definition 11.1.** A **variation of** ω (keeping endpoints fixed) is a function
> $$\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega$$
> for some $\varepsilon > 0$, such that:
>
> 1. $\bar{\alpha}(0) = \omega$;
> 2. There is a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ so that the map $\alpha : (-\varepsilon, \varepsilon) \times [0,1] \to M$ defined by $\alpha(u,t) = \bar{\alpha}(u)(t)$ is C^∞ on each strip $(-\varepsilon, \varepsilon) \times [t_{i-1}, t_i]$, $i = 1, \ldots, k$.

Since each $\bar{\alpha}(u)$ belongs to $\Omega = \Omega(M; p, q)$, note that $\alpha(u, 0) = p$, $\alpha(u, 1) = q$ for all $u \in (-\varepsilon, \varepsilon)$.

We will use either α or $\bar{\alpha}$ to refer to the variation. More generally if, in the above definition, $(-\varepsilon, \varepsilon)$ is replaced by a neighborhood U of 0 in $\mathbb{R}^n$, then α (or $\bar{\alpha}$) is called an n-parameter variation of ω. Now $\bar{\alpha}$ may be considered as a smooth path in Ω. Its velocity vector $\frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}u}\Big|_{u=0} \in T\Omega_\omega$ is defined to be the vector field W along ω given by:

$$W_t = \frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}u}(0)\Big|_t = \frac{\partial \alpha}{\partial u}(0, t)$$

Clearly $W \in T\Omega_\omega$. This vector field W is also called the **variation vector field** associated with the variation α.

Critical Paths

Definition 11.2. A path ω is a **critical path** for a function $F : \Omega \to \mathbb{R}$ if and only if $\frac{\mathrm{d}(F(\bar{\alpha}(u)))}{\mathrm{d}u}\Big|_{u=0}$ is zero for every variation $\bar{\alpha}$ of ω.

The Hessian of the Energy Functional

Continuing with the analogy developed in the preceding section, we now wish to define a bilinear functional

$$E_{**} : T\Omega_\gamma \times T\Omega_\gamma \to \mathbb{R}$$

when γ is a critical point of the function E, i.e., a geodesic. This bilinear functional will be called the **Hessian of E at γ**.

To define E_{**}, given vector fields $W_1, W_2 \in T\Omega_\gamma$ choose a 2-parameter variation

$$\alpha : U \times [0,1] \to M$$

where U is a neighborhood of $(0,0)$ in $\mathbb{R}^2$, so that

$$\alpha(0,0,t) = \gamma(t), \quad \frac{\partial \alpha}{\partial u_1}(0,0,t) = W_1(t), \quad \frac{\partial \alpha}{\partial u_2}(0,0,t) = W_2(t)$$

Then the Hessian $E_{**}(W_1, W_2)$ will be defined to be the second partial derivative

$$\frac{\partial^2 E(\bar{\alpha}(u_1,u_2))}{\partial u_1 \partial u_2}\Big|_{(0,0)}$$

where $\bar{\alpha}(u_1,u_2) \in \Omega$ denotes the path $\bar{\alpha}(u_1,u_2)(t) = \alpha(u_1,u_2,t)[cite:5]$. This second derivative will be written briefly as $\frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0)$.

Defintion: The index λ of the Hessian $E_{**} : T\Omega_\gamma \times T\Omega_\gamma \to \mathbb{R}$ is defined to be the maximum dimension of a subspace of $T\Omega_\gamma$ on which E_{**} is negative definite.

The Energy Functional and Geodesics on Path Spaces

Just like the height function on the torus in finite-dimensional Morse theory, we now want to choose a special function on the path space $\Omega(M)$ to develop our infinite-dimensional Morse theory. Natural candidates include the length of a path or the energy of a path, which map $\Omega(M) \to \mathbb{R}$.

However, to even define the length or energy of a path in M, we need a way to measure distances and speeds. Therefore, henceforth we will assume that M is a Riemannian manifold equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$.

For any piecewise smooth path $\omega \in \Omega(M)$, we define the energy functional $E : \Omega(M) \to \mathbb{R}$ by:

$$E(\omega) = \int_0^1 \left\| \frac{\mathrm{d}\omega}{\mathrm{d}t} \right\|^2 \mathrm{d}t = \int_0^1 \left\langle \frac{\mathrm{d}\omega}{\mathrm{d}t}, \frac{\mathrm{d}\omega}{\mathrm{d}t} \right\rangle \mathrm{d}t$$

Using the energy functional rather than arc length is a standard and analytically convenient choice in infinite-dimensional Morse theory because E is smooth, whereas the length functional suffers from lack of differentiability at constant paths.

The primary motivation for studying the energy functional on path spaces is its deep connection to differential geometry. The critical paths of E correspond exactly to the paths that satisfy the geodesic equations.

> **Theorem:** A path $\omega \in \Omega(M)$ is a critical point for the energy functional $E : \Omega(M) \to \mathbb{R}$ if and only if ω is a smooth geodesic connecting p to q.

The Morse Index and Finiteness

Analyzing the Hessian and the index of a critical path requires advanced techniques and tools from Riemannian geometry. In particular, there is a precise characterization of the index of a path using Jacobi fields and conjugate points, which we will not get into here. However, a fundamental result flowing from this geometric framework is formally captured by the following theorem:

> **Theorem (Finiteness of the Morse Index)** The Morse index λ of any critical point (geodesic) of the energy functional E on $\Omega(M)$ is always finite.

Finite-Dimensional Approximations of Path Spaces

The space Ω as a whole is infinite-dimensional. However, by restricting our attention to paths with bounded energy, we can construct finite-dimensional models that are also deformation retracts. Specifically, since a deformation retract preserves homotopy type, these finite-dimensional models share the exact same higher homotopy groups as the corresponding sublevel sets of Ω.

Construction of Broken Geodesic Subspaces

Given $c > 0$, let Ω^c denote the closed subset $E^{-1}([0, c]) \subset \Omega$ and let $\operatorname{Int} \Omega^c$ denote the open subset $E^{-1}([0, c))$ (where $E : \Omega \to \mathbb{R}$ is the energy functional) [cite: 5]. We study the topology of Ω^c by constructing a finite-dimensional approximation to it.

Choose a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of the unit interval. Let $\Omega(t_0, t_1, \ldots, t_k)$ be the subspace of Ω consisting of paths $\omega : [0, 1] \to M$ such that:

1. $\omega(0) = p$ and $\omega(1) = q$;
2. $\omega|_{[t_{i-1}, t_i]}$ is a geodesic for each $i = 1, \ldots, k$[cite: 5].

Finally, we define the restricted subspaces:

$$\Omega(t_0, t_1, \ldots, t_k)^c = \Omega^c \cap \Omega(t_0, t_1, \ldots, t_k)$$

$$\operatorname{Int} \Omega(t_0, t_1, \ldots, t_k)^c = (\operatorname{Int} \Omega^c) \cap \Omega(t_0, t_1, \ldots, t_k)$$

> **Lemma :** Let M be a complete Riemannian manifold, and let c be a fixed positive number such that $\Omega^c \neq \varnothing$. Then for all sufficiently fine subdivisions $(t_0, t_1, \ldots, t_k)$ of $[0, 1]$, the set $\operatorname{Int} \Omega(t_0, t_1, \ldots, t_k)^c$ can be given the structure of a smooth finite-dimensional manifold in a natural way.

To shorten notation, let us denote this manifold $\operatorname{Int} \Omega(t_0, t_1, \ldots, t_k)^c$ of broken geodesics by B. Let

$$E' : B \to \mathbb{R}$$

denote the restriction to B of the energy function $E : \Omega \to \mathbb{R}$.

> **Theorem :** This function $E' : B \to \mathbb{R}$ is smooth. Furthermore, for each $a < c$, the set $B^a = (E')^{-1}[0, a]$ is compact, and is a **deformation retract** of the corresponding set Ω^a. The critical points of E' are precisely the same as the critical points of E in $\operatorname{Int} \Omega^c$.

Because B^a is a deformation retract of Ω^a, the inclusion map $B^a \hookrightarrow \Omega^a$ is a homotopy equivalence. Consequently, they possess isomorphic homotopy groups:

$$\pi_k(B^a) \cong \pi_k(\Omega^a) \quad \text{for all } k \geq 0$$

This equivalence allows us to reduce the topology of infinite-dimensional path spaces directly to finite-dimensional Morse theory on B.

Stable Homotopy Groups of Spheres

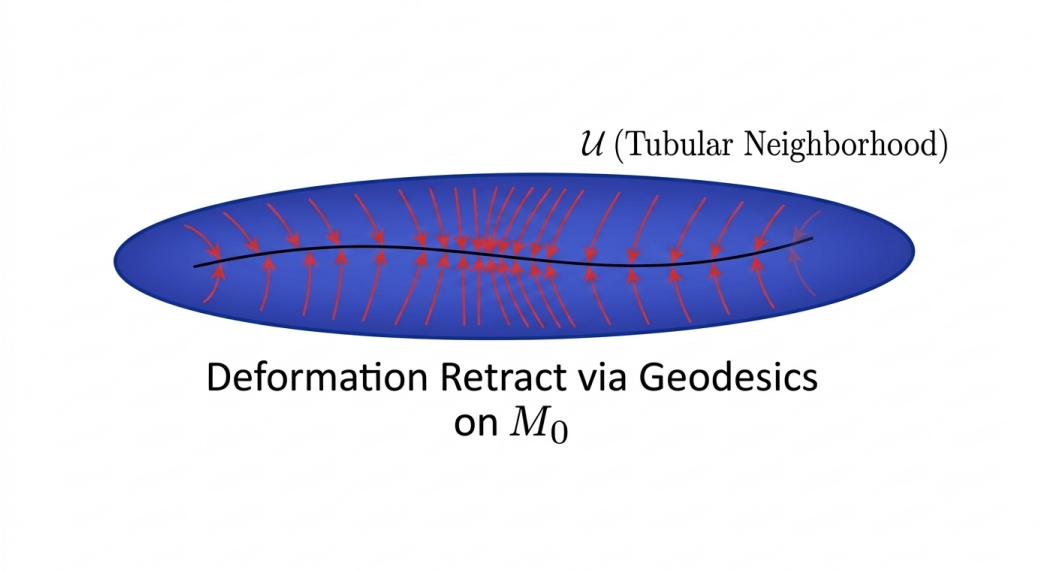
With all the machinery of Morse theory that we require finally in place, we will now give a proof of the fundamental stability property for the homotopy groups of spheres. Specifically, this states that the homotopy groups $\pi_{n+k}(S^n)$ are constant for all $n \geq k+2$, a classic result that follows directly from the Freudenthal Suspension Theorem which we will now prove.

Let $f : M \to \mathbb{R}$ be a smooth real-valued function with minimum 0, such that each sublevel set $M^c = f^{-1}[0, c]$ is compact.

> **Lemma :** If the set M^0 of minimal points is a manifold, and if every critical point >in $M - M^0$ has index $\geq \lambda_0$, then $\pi_r(M, M^0) = 0$ for $0 \leq r < \lambda_0$

Proof : First, observe that since M^0 is a smooth submanifold of M, we can invoke the Tubular Neighborhood Theorem. This guarantees the existence of a tubular neighborhood U of M^0 in M and a smooth retraction map $r : U \to M^0$ such that $r(x) = x$ for all $x \in M^0$, making M^0 a retract of U.

Furthermore, since M is a Riemannian manifold, we can shrink U appropriately so that for each point $u \in U$, its image $r(u) \in M^0$ and u itself are joined by a unique minimal geodesic (a standard property in Riemannian geometry). By sliding uniformly along these unique geodesics from u to $r(u)$, we obtain a smooth deformation retraction of U onto M^0. As the following picture below shows :



Now, let I^r denote the unit r-dimensional cube with boundary $\dot{I}^r$, where $0 \leq r < \lambda_0$, and let

$$h : (I^r, \dot{I}^r) \longrightarrow (M, M^0)$$

be any relative map. We wish to show that h is homotopic rel boundary to a map whose image lies entirely within M^0.

Let c be the maximum value of f on $h(I^r)$. Let $3\delta > 0$ be chosen such that it is smaller than the minimum of f on the closed complement $M - U$ (which exists because each sublevel set minus U is compact).

Next, we choose a smooth function $g : M^{c+2\delta} \to \mathbb{R}$ that approximates f extremely closely, satisfying:

1. $|f(x) - g(x)| < \delta$ for all $x \in M^{c+2\delta}$; and
2. The index of g at each critical point lying in the compact set $f^{-1}[\delta, c+2\delta]$ is $\geq \lambda_0$.

The function g is smooth on the compact region $g^{-1}[\delta, c+\delta] \subseteq f^{-1}[\delta, c+2\delta]$, and all its critical points are non-degenerate with index $\geq \lambda_0$. Consequently, the manifold $g^{-1}(-\infty, c+\delta]$ has the homotopy type of $g^{-1}(-\infty, 2\delta]$ with attached cells of dimension $\geq \lambda_0$ from the fundamental theorem of finite dimensional Morse Theory.

Now consider our map:

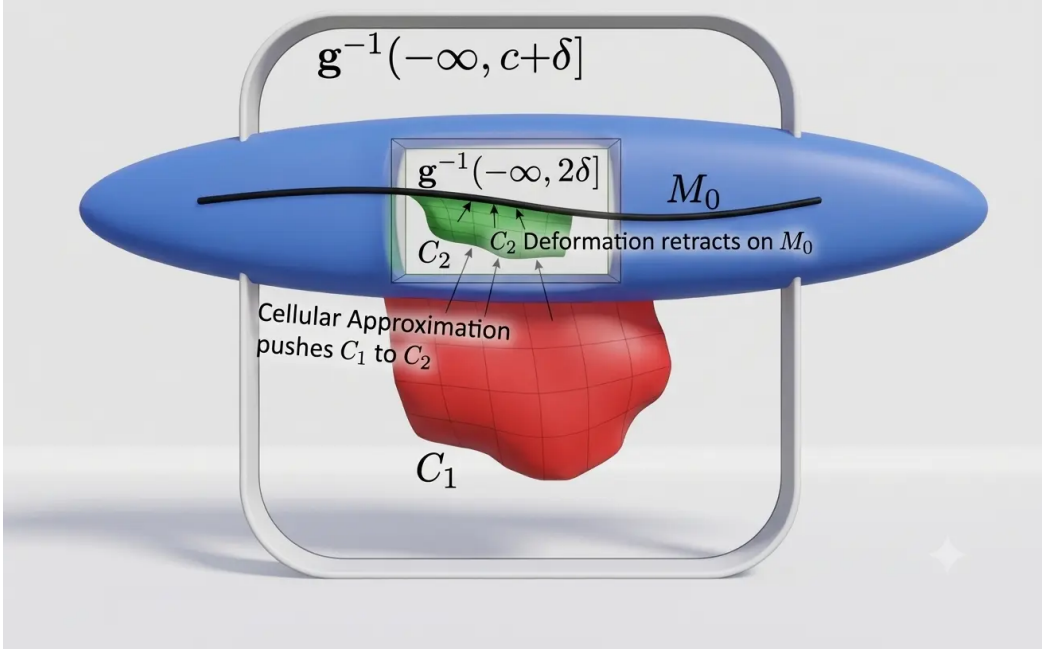
$$h : (I^r, \dot{I}^r) \longrightarrow (M^c, M^0) \subseteq (g^{-1}(-\infty, c+\delta], M^0)$$

Since the dimension of the domain r is strictly less than λ_0 ($r < \lambda_0$), and the manifold $g^{-1}(-\infty, c+\delta]$ is constructed by attaching cells of dimension $\geq \lambda_0$, it

follows from the **Cellular Approximation Theorem** that h is homotopic within $g^{-1}(-\infty, c+\delta]$ relative to M^0 to a map:

$$h' : (I^r, \dot{I}^r) \longrightarrow (g^{-1}(-\infty, 2\delta], M^0)$$

Finally, this image pair is contained entirely within (U, M^0). Since U deformation retracts onto M^0 via our Riemannian geodesic flow, h' is homotopic within (M, M^0) to a final map $h'' : (I^r, \dot{I}^r) \to (M^0, M^0)$, proving that $\pi_r(M, M^0) = 0$. The following diagram sums up what we have done in this proof.



Building on our foundation with the previous Lemma, we can now state the following theorem

> **Theorem :** If the space Ω^d of minimal geodesics from p to q is a topological manifold, and if every non-minimal geodesic from p to q has index $\geq \lambda_0$, then the relative homotopy group $\pi_i(\Omega, \Omega^d)$ is zero for $0 \leq i < \lambda_0$.

The proof proceeds by approximating the path space using finite-dimensional smooth manifolds. Clearly, it is sufficient to prove that

$$\pi_i(\text{Int } \Omega^c, \Omega^d) = 0$$

for arbitrarily large values of c~[?]. As established in earlier chapters, the space

$\text{Int } \Omega^c$ contains a smooth manifold $\text{Int } \Omega^c(t_0, \ldots, t_k)$ as a deformation retract, and the space Ω^d of minimal geodesics is fully contained within this smooth manifold.

The energy function $E : \Omega \to \mathbb{R}$, when restricted to $\text{Int } \Omega^c(t_0, \ldots, t_k)$, almost satisfies the hypotheses of Lemma. The only technical difficulty is that $E(m)$ ranges over the finite interval $[d, c)$ instead of the required interval $[0, \infty)$~[?]. To fix this, we can let

$$F : [d, c) \longrightarrow [0, \infty)$$

be any smooth diffeomorphism.

The composition $F \circ E : \text{Int } \Omega^c(t_0, \ldots, t_k) \to \mathbb{R}$ now fully satisfies the hypotheses of the previous Lemma. Hence, the relative homotopy groups satisfy:

$$\pi_i(\text{Int } \Omega^c(t_0, \ldots, t_k), \Omega^d) \cong \pi_i(\text{Int } \Omega^c, \Omega^d)$$

which is zero for $i < \lambda_0$. This completes the proof of Theorem.

Corollary:

We can derive a powerful structural consequence of this theorem by examining the long exact sequence of relative homotopy groups for the pair (Ω, Ω^d):

$$\cdots \longrightarrow \pi_i(\Omega^d) \xrightarrow{j_*} \pi_i(\Omega) \longrightarrow \pi_i(\Omega, \Omega^d) \longrightarrow \pi_{i-1}(\Omega^d) \longrightarrow \cdots$$

From Theorem, we know that $\pi_i(\Omega, \Omega^d) = 0$ for $i < \lambda_0$~[?]. Inspecting the exact sequence around these vanishing relative groups leads directly to our main takeaway:

> **Corollary :** $\pi_i(\Omega^d) \cong \pi_i(\Omega) \cong \pi_{i+1}(M)$

When M is the sphere S^n, the space of minimal geodesics Ω^d can be naturally identified with the equator of S^n and hence with S^n itself. The index of all geodesics from the south pole to the north pole is given by $k(n-1)$, with $n-1$ being the index of the smallest non-constant geodesic. Thus, our index bound parameter is $\lambda_0 = n - 1$. Applying our corollary to the case of the sphere yields the celebrated **Freudenthal Suspension Theorem**:

> **Freudenthal Suspension Theorem :** The suspension homomorphism
>
> $$\Sigma : \pi_i(S^n) \longrightarrow \pi_{i+1}(S^{n+1})$$
>
> is an isomorphism for $i \leq 2n - 3$.

Now, the following Stable property of the Homotopy groups is clear.

> **Stable Homotopy Property of Spheres :** $\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1})$ for $n \geq k+2$, establishing the stability of the homotopy groups of spheres in the stable range.

Computing Higher Homotopy Groups of Spheres

Having established the foundational tools of algebraic topology and infinite-dimensional path spaces, we now calculate specific higher homotopy groups of spheres. A fundamental vanishing result governs the dimensions where the sphere dimension exceeds the homotopy dimension:

> **Theorem :** For any integers n and k such that $n > k$, the k-th homotopy group of the n-dimensional sphere vanishes:
> $$\pi_k(S^n) = 0$$

This result follows directly from the stable range of homotopy groups of spheres (via the Freudenthal suspension theorem) combined with connectivity properties. Specifically, for $n > k$, the difference $n - k \geq 1$, which implies that $n - k + 1 \geq 2$. Consequently, the sphere S^{n-k+1} is simply connected, meaning its first homotopy group vanishes:

$$\pi_1(S^{n-k+1}) = 0$$

Through the structural stability and suspension isomorphisms, this vanishing propagates downward, yielding the general vanishing property for all dimensions below the sphere dimension.

Alternative Proof via Differential Topology

We can also verify this vanishing property using tools from differential topology and Sard's theorem, providing a geometric perspective on why higher-dimensional spheres cannot wrap around lower-dimensional ones non-trivially.

1. **Smooth Representatives and Sard's Theorem:** Let $f : S^k \to S^n$ be a smooth representative of an element in the homotopy group $[f] \in \pi_k(S^n)$. By the smooth approximation theorem, we can always assume f is smooth.

2. **Non-Surjectivity:** By Sard's Theorem (or standard measure theory on manifolds), since the dimension of the domain is strictly less than the dimension of the target ($k < n$), the image of f has a Lebesgue measure of zero in S^n. In particular, f cannot be surjective; there exists at least one point $y \in S^n$ that is not in the image of f, meaning $y \notin f(S^k)$.

3. **Stereographic Projection & Contractibility:** Because the image misses the point y, we can factor the map through the punctured sphere:

   $$f : S^k \to S^n \setminus \{y\}$$

   Using stereographic projection from y, the punctured sphere $S^n \setminus \{y\}$ is diffeomorphic to Euclidean space $\mathbb{R}^n$.

4. **Null-Homotopy:** Since Euclidean space $\mathbb{R}^n$ is contractible (it deformation retracts onto the origin), any continuous map into it is null-homotopic—meaning it can be continuously deformed to a constant map.

Therefore, any map $f : S^k \to S^n$ with $n > k$ is homotopic to a constant map, confirming the vanishing theorem.

0th Stable Homotopy Groups of Spheres

To study the homotopy groups of spheres further, it is essential to introduce a fundamental fiber bundle known as the **Hopf fibration**.

We can understand the origin of this bundle geometrically: $\mathbb{CP}^2$ can be thought of as attaching a 2 cell to $\mathbb{CP}^1$. Now, $\mathbb{CP}^1$ can be identified with S^2 (or viewed as a quotient space of S^3 whose orbits are circles), and the boundary of the attaching cell is S^3, which maps to this to $\mathbb{CP}^1$. By viewing $\mathbb{CP}^1$ as S^2, we obtain a natural map from S^3 to S^2 whose fibers are circles S^1. A little bit of work shows that this map is locally trivial, yielding the fiber bundle:

$$S^1 \to S^3 \to S^2$$

The Long Exact Sequence and Homotopy Calculation

From the machinery of algebraic topology established earlier, since the base space S^2 is paracompact and the projection is a locally trivial fiber bundle, we obtain a long exact sequence of homotopy groups:

$$\cdots \to \pi_i(S^1) \to \pi_i(S^3) \to \pi_i(S^2) \to \pi_{i-1}(S^1) \to \dots$$

Let us extract the relevant chunk of this exact sequence for lower dimensions:

$$\cdots \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3) \to \dots$$

We know from our earlier results that the homotopy groups of the 3-sphere in these dimensions vanish, specifically:

$$\pi_1(S^3) = 0 \quad \text{and} \quad \pi_2(S^3) = 0$$

Substituting these vanishing terms into our exact sequence yields:

$$0 \to \pi_2(S^2) \to \pi_1(S^1) \to 0$$

By exactness, the homomorphism between $\pi_2(S^2)$ and $\pi_1(S^1)$ is an isomorphism. Since the fundamental group of the circle is known to be the integers, $\pi_1(S^1) \cong \mathbb{Z}$, we immediately conclude that:

$$\pi_2(S^2) \cong \pi_1(S^1) \cong \mathbb{Z}$$

From stable homotopy theory, we know that the homotopy groups $\pi_{n+k}(S^n)$ stabilize for large n (specifically in the stable range where $n \geq k+2$). In particular, the homotopy groups $\pi_n(S^n)$ stabilize starting from $\pi_2(S^2)$. Since we have just calculated that $\pi_2(S^2) \cong \mathbb{Z}$, this stabilization property yields:

$$\pi_n(S^n) \cong \mathbb{Z} \quad \text{for all } n \geq 1$$

This gives us the **0th stable homotopy group of spheres**.

Homotopy Groups of Spheres via the Pontryagin-Thom Construction

The **Pontryagin-Thom construction** allows us to view the homotopy groups of spheres through a completely geometric lens—translating abstract homotopy classes of maps into tangible submanifolds equipped with normal bundle trivializations.

Framed Submanifolds and Framed Cobordisms

Before establishing the main correspondence, we formalize the notions of framed submanifolds and framed cobordisms.

Definition: Framed Submanifold

Let M be a smooth, connected N-dimensional manifold, and let $V^k \subset M$ be a closed, smooth k-dimensional submanifold. The normal bundle $\nu(V)$ of V in M is an $(N-k)$-dimensional vector bundle over V.

A **framing** of V^k in M is a choice of $N-k$ linearly independent sections $F = (v_1, v_2, \ldots, v_{N-k})$ of the normal bundle $\nu(V)$ that span the normal space T_xM/T_xV at every point $x \in V$. A pair (V^k, F) consisting of a submanifold together with a choice of normal framing is called a **framed submanifold** of M.

$$\text{Normal Framing: } \nu(V) \cong V \times \mathbb{R}^{N-k}$$

Definition: Framed Cobordism

Let (V_0^k, F_0) and (V_1^k, F_1) be two k-dimensional framed submanifolds of M. We say (V_0, F_0) and (V_1, F_1) are **framed cobordant** if there exists a smooth, compact $(k+1)$-dimensional submanifold with boundary

$$W^{k+1} \subset M \times [0,1]$$

such that:

1. The boundary satisfies $\partial W = (V_0 \times \{0\}) \sqcup (V_1 \times \{1\})$.
2. W intersects the boundary components $M \times \{0\}$ and $M \times \{1\}$ orthogonally.
3. W carries a normal framing F_W in $M \times [0,1]$ that restricts to F_0 on $V_0 \times \{0\}$ and to F_1 on $V_1 \times \{1\}$.

Framed cobordism forms an equivalence relation. The set of equivalence classes of k-dimensional framed submanifolds in M is denoted by $\Omega_k^{\text{framed}}(M)$.

The fundamental insight of Lev Pontryagin and René Thom is that homotopy classes of maps into a sphere correspond bijectively to framed cobordism classes of submanifolds.

> **Theorem (Pontryagin-Thom Theorem) :** For all integers $n \geq 1$ and $k \geq 0$, there is a natural bijection between the homotopy classes of maps from S^{n+k} to S^n and the framed cobordism classes of **closed**, k-dimensional submanifolds of S^{n+k}
>
> $$\pi_{n+k}(S^n) \cong \Omega_k^{\text{framed}}(S^{n+k})$$

Visualizing the 1st Stable Homotopy Group: $\pi_4(S^3) \cong \mathbb{Z}_2$

Equipped with the Pontryagin-Thom construction, we can now compute a non-trivial stable homotopy group of spheres completely geometrically, although it mainly just gives us intuition, it is not strictly a proof. Specifically, we wish to calculate $\pi_4(S^3)$, which corresponds via stabilization to the 1st stable homotopy group π_1^s.

By the Pontryagin-Thom theorem, this is isomorphic to the group of 1-dimensional framed cobordism classes in S^4:

$$\pi_4(S^3) \cong \Omega_1^{\text{framed}}(S^4)$$

By the classification of compact, connected 1-dimensional manifolds without boundary, any such submanifold is diffeomorphic to the circle S^1. Thus, we only need to study framed circles embedded in S^4. To classify them up to framed cobordism, we divide the analysis into two natural cases: different framings on the same submanifold, and different submanifolds altogether.

Case I: Different Framings on the Same Submanifold

Suppose we fix a single embedded circle $M \cong S^1 \subset S^4$.

1. **Orthogonal Framings:** By applying the Gram-Schmidt orthogonalization process to any normal framing, we can assume WLOG that the framing consists of orthonormal vectors at every point along M.

2. **Identification with $SO(3)$:** The normal bundle $\nu(M)$ of M in S^4 has rank $4-1=3$. At each point $x \in M$, the framing is an ordered orthonormal basis of the 3-dimensional normal space $\nu_x M \cong \mathbb{R}^3$. This identifies each point's framing with an element of the special orthogonal group $SO(3)$.

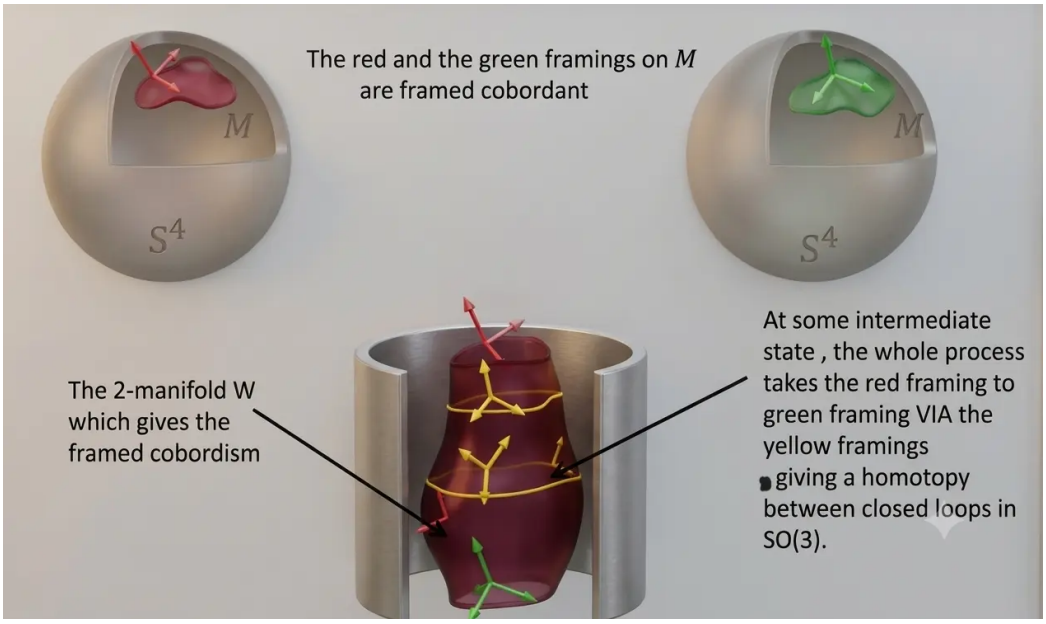
3. **Loops in $SO(3)$:** As we trace once around the circle M, the framing rotates, defining a continuous loop in $SO(3)$:

$$\gamma : S^1 \to SO(3)$$

4. **Framed Cobordism and Homotopy:** Two framings on M are framed cobordant if and only if the cobordism (a 2-manifold W spanning between them in $S^4 \times [0,1]$) provides a smooth extension of the framing. This precisely transcends down to a **free homotopy** between the corresponding loops in $SO(3)$ as shown in the picture below.

5. **The Classification:** Since the fundamental group of the rotation group is $\pi_1(SO(3)) \cong \mathbb{Z}_2$, there are precisely two free homotopy classes of loops in $SO(3)$ (the trivial class and the non-trivial twisted class).

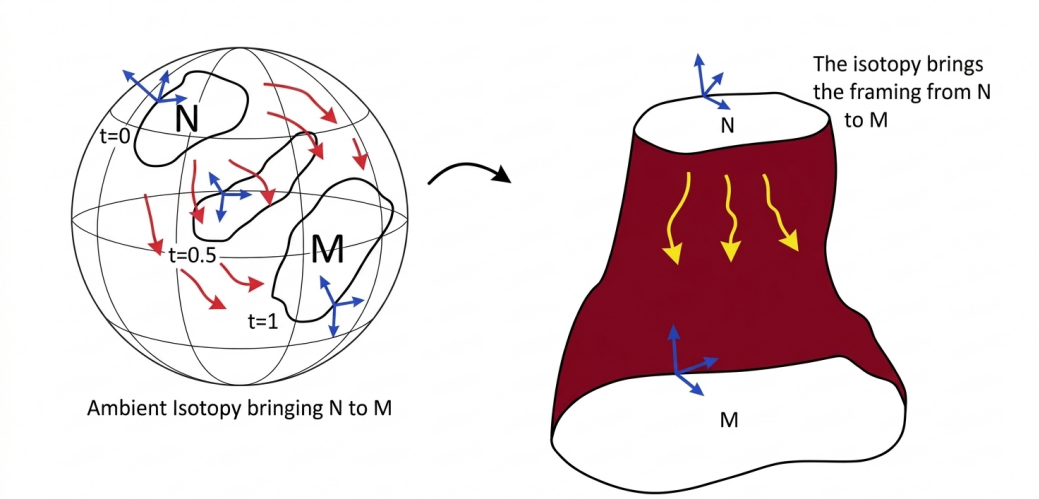
Thus, up to framed cobordism, there are exactly two distinct framings that can be placed on a fixed 1-dimensional submanifold M.



Case II: Different Submanifolds in S^4

Suppose we have two different embedded circles, M and N, in S^4.

1. **Ambient Isotopy:** M and N are ambiently isotopic because of the fact that all S^1 knots in S^4 are trivial. There exists a smooth family of diffeomorphisms of S^4 moving M directly onto N.
2. **Pulling Back Framings:** We can use this ambient isotopy to push any framed submanifold N back onto M, pulling back its framing without changing its framed cobordism class as shown in the picture below, which reduces all cases back to **Case I**.



Combining both cases, the classification of 1-dimensional framed submanifolds in S^4 up to cobordism reduces entirely to the homotopy classes of loops in $SO(3)$. Since there are exactly two such classes, we arrive at the celebrated calculation:

$$\pi_4(S^3) \cong \Omega_1^{\text{framed}}(S^4) \cong \mathbb{Z}_2$$

It should be noted that this is not strictly a proof but his geometric approach visually explains why the 1st stable homotopy group of spheres collapses to $\mathbb{Z}_2$.

Conclusion :

Morse theory allows us to obtain cell decompositions of a manifold, while path spaces help us get a foothold on higher homotopy groups. Combined with foundational homotopy theory results, the Morse theory of M and its path spaces provides a powerful machinery for establishing the stability of higher homotopy groups.

Toward the end, we explored the Pontryagin-Thom construction—a deeply geometric lens through which to view homotopy groups. In principle, one can through the Pontryagin-Thom theorem prove the Stable Homotopy of Spheres,

making the underlying intuition far more transparent. However, doing so requires careful general position arguments from differential topology, such as the Thom-Jet transversality theorems, which we bypassed here.

Nevertheless, this perspective lets us catch a glimpse of the first stable homotopy group, a structure that is far from obvious using only homotopy exact sequences unless you bring out heavy artillery like the Serre or Adams spectral sequences.

Ultimately, the grand takeaway is this: to prove a general topological result, it is often vastly more fruitful to smooth it out and tackle it in the smooth category, where a much richer arsenal of geometric and analytic techniques is at your disposal.