Definitions and First Examples

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In linear algebra, we have learned that an F-**algebra** is simply a vector space V over F equipped with an additional product structure that is **bilinear**, i.e. a binary operation $V \times V \to V$ where $(x, y) \mapsto x \cdot y$ such that

$$(\lambda_1 x_1 + \lambda_2 x_2) \cdot y = \lambda_1(x_1 \cdot y) + \lambda_2(x_2 \cdot y)$$

$$x \cdot (\lambda_1 y_1 + \lambda_2 y_2) = \lambda_1(x \cdot y_1) + \lambda_2(x \cdot y_2)$$

The Lie algebra is a special kind of algebra with additional requirements, and it can be seen as a generalization of the following important **commutator** operation $[x, y] = xy - yx$.

Remark 0.1. *Why is this called the commutator? Because we see that $[x, y] = 0$ implies that $xy = yx$ by definition, meaning that x and y commutes under the product.*

The notion of Lie algebra

Definition 0.1: Lie Algebra

A vector space L (standing for "Lie algebra") over a field F, with an operation $L \times L \to L$, denoted by $(x, y) \mapsto [xy]$ (or sometimes $[x, y]$) called the **bracket** or **commutator** of x and y is called a **Lie algebra** over F if the following axioms are satisfied

- (L1) The bracket operation is bilinear.
- (L2) $[xx] = 0$ for all $x \in L$.
- (L3) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ $(x, y.z \in L)$.

Here (L3) is called the **Jacobi Identity**.

Remark 0.2. *It is very important to note that the bracket operation is **neither commutative nor associative**.*

We have the following immediate consequence from the definition

Proposition 0.1 (The Lie bracket is anticommutative)**.** *Let L be a Lie algebra over F. For any $x, y \in L$,*

$$[xy] = -[yx]$$

Proof. Applying (L1) to the bracket $[x + y, x + y]$, we have:

$$
\begin{aligned}
[x+y, x+y] &= [x, x+y] + [y, x+y] \\
&= [x,x] + [x,y] + [y,x] + [y,y] \\
&= [x,y] + [y,x]
\end{aligned}
$$

Using (L2), $[x+y, x+y] = 0$, so we have
$[xy] + [yx] = 0 \implies [xy] = -[yx]$. □

In fact, if $\text{char}(F) \neq 2$, then anti-commutativity implies (L2) because then

$$[xx] = -[xx] \implies 2[xx] = 0 \implies [xx] = 0.$$

Here we need the assumption that $\text{char}(F) \neq 2$ in the final implication since we need $2 \neq 0$ to conclude that $[xx] = 0$.

Example 0.1: $(\mathbb{R}^3, \times)$ is a Lie algebra

Recall that the cross-product in $\mathbb{R}^3$ is defined by

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{pmatrix} y_1 z_2 - y_2 z_1 \\ z_1 x_2 - z_2 x_1 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

By direct computation, we may verify that this defines a bracket operation satisfying (L1). (L2) and (L3). This is not common in vector spaces of arbitrary dimensions because the construction of the cross-product is highly dependent on the structure of $\mathbb{R}^3$. (cf. Exercise 1)

As with all abstract structures in mathematics, it is very important to study the structure of a newly-defined algebraic object. There are mainly two ways we study the structure

1. We study "**structure-preserving**" maps in between them to understand whether two objects are essentially the same, relating complicated-seeming objects to simpler ones.

2. We study the **sub-objects** to understand the internal structure and hierarchy within the object.

These two ideas leads to the following notions for Lie algebras:

Definition 0.2: Isomorphism of Lie Algebras

Two Lie algebras L, L' over F are said to be **isomorphic** if there exists an linear isomorphism $\phi : L \to L'$ that preserves the bracket operation, i.e.

$$\phi([xy]) = [\phi(x)\phi(y)]$$

for all $x, y \in L$. Such a ϕ is called an **isomorphism of Lie algebras**. More generally, if ϕ is not necessarily bijective, then it is called a **homomorphism of Lie algebras**.

Definition 0.3: Subalgebra

A linear subspace K of a Lie algebra L is called a (Lie) **subalgebra** if $[xy] \in K$ for all $x, y \in K$, i.e. it is closed under the bracket operation.

Remark 0.3. *Any subalgebra K of L is itself a Lie algebra with its operations inherited from L.*

We shall begin with the following simplest exmaple of Lie algebra.

Example 0.2: One-dimensional subalgebra

For any Lie algebra L over F and $x \in L$, we have the one-dimensional Lie algebra

$$Fx = \{ax : a \in F\}.$$

Clearly $0 \in Fx$ and it is closed under bracket operation as for any $a_1, a_2 \in F$:

$$[a_1x, a_2x] = a_1a_2[x, x] = 0 \in Fx$$

by (L2). The above computation also shows that the bracket operation on the one-dimensional Lie algebra Fx is **trivial**, or in other words, it is an **abelian Lie algebra**.

Remark 0.4. *In the rest of this note, we shall assume that L is a* ***finite-dimensional*** *Lie algebra over F.*

Linear Lie Algebras

In this subsection, we introduce a family of important, **non-trivial** examples of Lie algebras – the **linear Lie algebras**. The moral of the construction is the following: while it is difficult to define a bracket operation directly in the vector spaces themselves, it is much easier to define a Lie algebra structure on the linear maps from a vector space to itself. Therefore, we consider the following vector space:

Definition 0.4: Endomorphisms

Let V be a finite-dimensional vector space over F. An **endomorphism** of V is a linear map $f : V \to V$. We denote

$$\operatorname{End} V := \{f : V \to V | f \text{ is linear}\}$$

We may verify that $\operatorname{End} V$ is itself a vector space with addition and scalar multiplication defined component-wise. With a choice of basis of V, we may represent every element of $\operatorname{End} V$ as an $n \times n$ matrix, assuming that $\dim V = n$. We may treat each matrix as a vector of length n^2, then a basis for $\operatorname{End} V$ would simply be the matrices with only a 1 in a particular entry, and zero in all others, just like the standard orthonormal basis of $\mathbb{R}^n$. More precisely, define

$$e_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \underset{(i,j)}{1} & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

then $\{e_{ij} : i = 1, ..., n;\ j = 1, ..., n\}$ is a basis for $\operatorname{End} V$, so $\dim \operatorname{End} V = n^2$.

We notice that matrix multiplication / composition of linear maps gives a natural product structure on $\operatorname{End} V$, thereby making $\operatorname{End} V$ an F-algebra. Moreover, this product structure also gives rise to a natural bracket operation on $\operatorname{End} V$, as we have hinted at the beginning of the chapter:

Proposition 0.2 (Commutator on $\operatorname{End} V$ defines a Lie algebra). *For any $x, y \in \operatorname{End} V$, define $[x, y] = xy - yx$, then $(\operatorname{End} V, [,])$ is a Lie algebra.*

Proof. It suffices to verify axioms (L1), (L2), (L3).

- (L1) For any $\lambda_1, \lambda_2 \in F$, $x_1, x_2 \in \operatorname{End} V$:

$$\begin{aligned}[\lambda_1 x_1 + \lambda_2 x_2, y] &= (\lambda_1 x_1 + \lambda_2 x_2)y - y(\lambda_1 x_1 + \lambda_2 x_2) \\ &= \lambda_1(x_1 y - y x_1) + \lambda_2(x_2 y - y x_2) \\ &= \lambda_1[x_1, y] + \lambda_2[x_2, y]\end{aligned}$$

  and similarly for the y component.

- (L2) This is trivial as $xx - xx = 0$.

- (L3) We verify the Jacobi Identity. We first compute

$$[x[yz]] = [x(yz - zy)] = [x(yz)] - [x(zy)] = xyz - yzx - xzy + zyx$$

  Similarly we have:

$$[y[zx]] = yzx - zxy - yxz + xzy$$

$$[z[xy]] = zxy - xyz - zyx + yxz$$

and adding the three together, we see that all terms cancel, and we are left with 0, thus verifying the Jacobi identity.

We thereby have the notion of linear Lie algebras □

Definition 0.5: Linear Lie Algebras

Let V be a finite-dimensional vector space over F, then the Lie algebra $\operatorname{End} V$ equipped with the commutator operation as above is called the **general linear algebra**, denoted by $\mathfrak{gl}(V)$. Moreover, any subalgebra of the Lie algebra $\mathfrak{gl}(V)$ is called a **linear Lie algebra**.

Remark 0.5. *If we fix a basis of V, then we can regard $\mathfrak{gl}(V)$ as the set of all $n \times n$ matrices over F. In this case, we denote it as $\mathfrak{gl}(n, F)$.*

Next, we introduce four families of examples of linear Lie algebras: A_l, B_l, C_l, D_l for $l \geq 1$, called the **classical algebras**, that will be highly important for the development of the theory.

Example 0.3: A_l

Let $\dim V = l + 1$, denote by $\mathfrak{sl}(V)$ or $\mathfrak{sl}(l+1, V)$, the set of endomorphism of v having trace zero. Using the fact that $\operatorname{Tr}(xy) = \operatorname{Tr}(yx)$ and $\operatorname{Tr}(x + y) = \operatorname{Tr}(x) + \operatorname{Tr}(y)$, we see that

$$\operatorname{Tr}([x, y]) = \operatorname{Tr}(xy - yx) = \operatorname{Tr}(xy) - \operatorname{Tr}(yx) = 0$$

so indeed $\mathfrak{sl}(V)$ is a linear subspace of $\mathfrak{gl}(V)$ that is closed under the bracket operation and is thus a subalgebra of $\mathfrak{gl}(V)$. This linear Lie algebra $\mathfrak{sl}(V)$ is called the **special linear algebra**. We use A_l to denote the class of all special linear algebras.

Proposition 0.3. *The special linear algebra $\mathfrak{sl}(l + 1, F)$ has dimension $(l + 1)^2 - 1$.*

Proof. Firstly, we notice that $\mathfrak{sl}(V)$ is always a **proper** subalgebra of $\mathfrak{gl}(V)$, as $\mathfrak{gl}(V)$ contains endomorphism with non-zero trace. Hence $\dim \mathfrak{sl}(V)$ is at most $(l + 1)^2 - 1$. Next, it suffices to find a linearly independent set consisting of $(l + 1)^2 - 1$ elements. For this purpose, we choose e_{ij} where $i \neq j$ and $h_i = e_{ii} - e_{i+1,i+1}$. This is clearly linearly independent, and its length is:

$$\underbrace{((l + 1)^2 - l)}_{\#\{e_{ij}(i \neq j)\}} + \underbrace{((l + 1) - 1)}_{\#\{e_{ii} - e_{i+1,i+1}\}} = (l + 1)^2 - 1$$

so we must have that $\dim \mathfrak{sl}(l + 1, F) = (l + 1)^2 - 1$, and the above chosen basis will always be regarded as the standard basis for $\mathfrak{sl}(n + 1, F)$. □

Remark 0.6. *How do we understand the choice of basis above? What we are doing is actually decomposing $\mathfrak{sl}(l+1,F)$ into the following direct sum:*

$$\mathfrak{sl}(l+1,F) = \underbrace{\{(a_{ij}) \in \mathfrak{sl}(l+1,F) : a_{ii} = 0\}}_{\text{matrices with zero-diagonal}} \oplus \underbrace{\{(a_{ij}) \in \mathfrak{sl}(l+1,F) : a_{ij} = 0(i \neq}_{\text{matrices with zero off-diagonal entrie}}$$

In other words, we are looking at the off-diagonal and diagonal entries separately. Notice that the off-diagonal entries do not affect the trace, so we can freely choose then as in the situation of $\mathfrak{gl}(l+1,F)$. Consequently, $\{e_{ij} : i \neq j\}$ is a basis of A_1. Then we consider the diagonal entries. We notice that $e_{ii} - e_{i+1,i+1}$ all have trace zero, and together they form a linearly independent set in A_2, which is a proper subspace of the space of all diagonal matrices. A similar counting argument as in the proof above then shows that this indeed is a basis of A_2. Therefore, putting all of them together, we get a basis of $\mathfrak{sl}(l+1,F)$.

To illustrate the basis more concretely, we write down the basis for $\mathfrak{sp}(2,F)$:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The reader is encouraged to write down the standard basis for $\mathfrak{sp}(3,F)$ given in the proof above. The above proof hints at the general method in finding a standard basis for linear Lie algebras:

1. Split matrices into blocks in a suitable fashion.
2. Compute to find restrictions on each block.
3. Find a basis for each block under the restriction, then put them together to form a basis of the entire algebra.

Example 0.4: C_l

Let $\dim V = 2l$ with a basis $(v_1, ..., v_{2l})$. Let f be the non-degenerate skew-symmetric form represented by the matrix:

$$s = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}.$$

In other words, $f : V \times V \to V$ is the non-degenerate bilinear form defined by: $f(v, w) = v^T sw$. Denote by $\mathfrak{sp}(V)$ or $\mathfrak{sp}(2l, F)$, the **symplectic algebra**, which consists of all endomorphisms $x \in \operatorname{End}(V)$ such that

$$f(x(v), w) = -f(v, x(w))$$

for any $v, w \in V$. (Notice the similarity with self-adjoint matrices.) We can verify that $\mathfrak{sp}(V)$ is indeed closed under the bracket operation: for any $x_1, x_2 \in \operatorname{End} V$,

$$\begin{aligned} f([x_1x_2](v), w) &= f((x_1x_2 - x_2x_2)v, w) \\ &= f((x_1x_2)v, w) - f((x_2x_1)v, w) \\ &= f(x_2(v), x_1(w)) - f(x_1(v), x_2(w)) \\ &= f(v, (x_2x_1)w) - f(v, (x_1x_2)w) \\ &= -f(v, [x_1x_2]w) \end{aligned}$$

so $\mathfrak{sp}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$. From this computation, we also see that the minus sign in the definition is necessary for us to define a subalgebra.

Proposition 0.4. *The symplectic linear algebra $\mathfrak{sp}(2l, F)$ has dimension $2l^2 + l$.*

Proof. We adopt the same methodology as above. For any $x \in \mathfrak{sp}(2l, F)$, we split x into blocks in the same way as s, i.e.

$$x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

where $m, n, p, q \in \mathfrak{gl}(V)$. We then notice that:

$$f(x(v), w) = -f(v, x(w)) \implies v^T x^T sw = -v^T sxw \implies x^T s = -sx$$

and using the blocks

$$x^T s = \begin{pmatrix} m^T & p^T \\ n^T & q^T \end{pmatrix} \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix} = \begin{pmatrix} -p^T & m^T \\ -q^T & n^T \end{pmatrix}$$

$$sx = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix} \begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} p & q \\ -m & -n \end{pmatrix}$$

Comparing each entry, we then see that

$$p^T = p, \quad n^T = n, \quad q^T = -m$$

are restriction on each block matrix. We find a basis for each block separately. For p, notice that the diagonal entries are free of choice, and for the off-diagonal entries, as long as the entry in the lower-left triangular region (a_{ij} where $j < i$) is chosen, then the reflection across the diagonal is determined. This is because p must be a symmetric matrix. Hence, a basis for p would be $e_{i,l+i}$ and $e_{i,l+j} + e_{j,l+i}$ for $1 \leq i < j \leq l$. Similarly for n. For m, we see that as long as the entries of m are determined, q will be determined by the relation $q^T = -m$, so together, the q-m diagonal blocks is generated by $e_{ij} - e_{l+j,l+i}$ where $1 \leq i \neq j \leq l$. Putting these together gives a basis of $\mathfrak{sp}(2l, F)$, and counting gives

$$\dim \mathfrak{sp}(2l, F) = \underbrace{\left(l^2 - \frac{1}{2}(l^2 - l)\right)}_{\#(\text{basis of } n)} + \underbrace{\left(l^2 - \frac{1}{2}(l^2 - l)\right)}_{\#(\text{basis of } p)} + \underbrace{l^2}_{\#(\text{basis of } m \text{ and } q)} = 2l^2 +$$

□

It is noticeable that based on the construction above, $\mathfrak{sp}(2, F)$ has the same standard basis as $\mathfrak{sl}(2, F)$, given above by (x, h, y). This implies that $\mathfrak{sp}(2, F)$ is isomorphic to $\mathfrak{sl}(2, F)$.

Example 0.5: B_l and D_l

Let $\dim V = 2l + 1$ be odd, and let f be the non-degenerate symmetric bilinear form on V represented by the matrix

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$$

The **orthogonal algebra**, denoted by $\mathfrak{o}(V)$ or $\mathfrak{o}(2l+1, F)$, consists of all endomorphisms of V such that $f(x(v), w) = -f(v, x(w))$ (the same condition as in $\mathfrak{sp}(V)$) holds. The same calculation as in example 0.4 shows that $\mathfrak{o}(2l+1, F)$ is indeed a linear Lie algebra. Applying the same methodology as in the proof of Proposition 0.4 shows that $\dim \mathfrak{o}(2l + 1, F) = 2l^2 + l$, which is the same as $\dim \mathfrak{sp}(2l, F)$. For the specific basis, refer to Humphreys. The set of all $\mathfrak{o}(2l+1, F)$ is denoted B_l.

When $\dim V = 2l$ is even, replace f above by the symmetric bilinear form represented by

$$s = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$$

we then obtain another orthogonal algebra $\mathfrak{o}(2l, F)$ with identical construction as $\mathfrak{o}(2l + 1, F)$. The set of all $\mathfrak{o}(2l, F)$ is denoted D_l. Similar calculations also shows that $\dim \mathfrak{o}(2l, F) = 2l^2 - l$.

Finally, we include some other subalgebras of $\mathfrak{gl}(n, F)$ which are also important but do not belong to the classical algebras.

Example 0.6: $\mathfrak{t}, \mathfrak{n}, \mathfrak{d}$

Let $\mathfrak{t}(n, F)$ be the set of **upper triangular matrices** (a_{ij}) where $a_{ij} = 0$ when $i > j$. Let $\mathfrak{n}(n, F)$ be the set of **strictly upper triangular matrices** consisting of upper triangular matrices with zero diagonal. Let $\mathfrak{d}(n, F)$ be the set of all diagonal matrices. As each of these sets are closed under the matrix product, they must be closed under the bracket operation and hence they are subalgebras of $\mathfrak{gl}(n, F)$. It is also evident that $\mathfrak{t}(n, F) = \mathfrak{n}(n, F) + \mathfrak{d}(n, F)$.

Notation: If H, K are subalgebras of a L, we use $[HK]$ or $[H, K]$ to denote the subspace of L spanned by the commutators $[xy]$ where $x \in H, y \in K$.

Proposition 0.5 (Derived Algebra of $\mathfrak{t}(n, F)$)**.** $[\mathfrak{t}(n, F), \mathfrak{t}(n, F)] = \mathfrak{n}(n, F)$.

Proof. We first show that $[\mathfrak{d}(n, F), \mathfrak{n}(n, F)] = \mathfrak{n}(n, F)$. It is trivial to see that the LHS is contained in the RHS. For any $A \in \mathfrak{n}(n, F)$, Let A_i be the matrix whose ith row is the same as the ith row of A, and the rest are all 0. We then have that

$$[e_{ii}A_i] = A_i$$

and consequently:

$$A = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} [e_{ii}A_i] \in [\mathfrak{d}(n, F), \mathfrak{n}(n, F)]$$

which shows that the RHS is contained in the LHS. We then show that $[\mathfrak{n}(n, F), \mathfrak{n}(n, F)] = \mathfrak{n}(n, F)$. Again, LHS is contained in the RHS, and moreover, we see that

$$A = \sum_{i<j} e_{ij}A = \sum_{i<j} [e_{ij}, e_{ij}A] \in [\mathfrak{n}(n, F), \mathfrak{n}(n, F)]$$

which proves the equality. Using both identities, we then see that

$$\begin{aligned}
[\mathfrak{t}(n, F), \mathfrak{t}(n, F)] &= [\mathfrak{n}(n, F) + \mathfrak{d}(n, F), \mathfrak{n}(n, F) + \mathfrak{d}(n, F)] \\
&= [\mathfrak{d}(n, F), \mathfrak{n}(n, F)] + [\mathfrak{n}(n, F), \mathfrak{d}(n, F)] + [\mathfrak{n}(n, F), \mathfrak{n}(n, F)] \\
&= \mathfrak{n}(n, F) + \mathfrak{n}(n, F) + \mathfrak{n}(n, F) \\
&= \mathfrak{n}(n, F)
\end{aligned}$$

which completes the proof. (cf. Exercise 5). $\square$

Lie Algebras of Derivations and the Adjoint Representation

Linear Lie algebras are subalgebras of $\operatorname{End} V$, where V is an arbitrary finite dimensional vector space. We get another important example of Lie algebra if we further require V to be an F-algebra itself:

Definition 0.6: Derivations

Let $\mathfrak{V}$ be an F-algebra endowed with a product structure. By a **derivation** of $\mathfrak{V}$, we mean a linear map $\delta \in \operatorname{End}\mathfrak{V}$ such that the product rule:

$$\delta(ab) = a\delta(b) + \delta(a)b$$

holds. The collection of all derivations of $\mathfrak{V}$ is denoted $\operatorname{Der}\mathfrak{V}$.

The important fact about $\operatorname{Der}\mathfrak{V}$ is that it is itself a Lie algebra with the bracket operation inherited from $\mathfrak{gl}(\mathfrak{V})$:

Proposition 0.6. $\operatorname{Der}\mathfrak{V}$ *is a subalgebra of* $\mathfrak{gl}(\mathfrak{V})$.

Proof. It is easy to verify that $\operatorname{Der}\mathfrak{V}$ is a linear subspace of $\mathfrak{gl}(\mathfrak{V})$. We shall only verify that it is closed under the bracket operation: for any $\delta, \delta' \in \operatorname{Der}\mathfrak{V}$, we want to show that $[\delta, \delta'] = \delta\delta' - \delta'\delta \in \operatorname{Der}\mathfrak{V}$. We have:

$$\begin{aligned}
[\delta, \delta'](ab) &= \delta(\delta'(ab)) - \delta'(\delta(ab)) \\
&= \delta(a\delta'(b) + \delta'(a)b) - \delta'(a\delta(b) + \delta(a)b) \\
&= a\delta\delta'(b) + \delta(a)\delta'(b) + \delta'(a)\delta(b) + \delta\delta'(a)b \\
&\quad - a\delta'\delta(b) - \delta'(a)\delta(b) - \delta(a)\delta'(b) - \delta'\delta(a)b \\
&= a(\delta\delta')(b) + (\delta\delta'(a))b - a(\delta'\delta)(b) - (\delta'\delta)(a)b \\
&= a[\delta, \delta'](b) + [\delta, \delta'](a)b
\end{aligned}$$

which completes the proof. □

Remark 0.7. *The regular product of derivations may not be another derivation, as we will see in the examples that follow.*

As a Lie algebra is itself an F-algebra by definition, the Lie algebra of derivations $\operatorname{Der} L$ is well-defined, and perhaps the most important elements in $\operatorname{Der} L$ are following:

Definition 0.7: Adjoint Representation

For $x \in L$, the map sending $y \mapsto [xy]$ is an endomorphism using bilinearity. We denote this map as $\operatorname{ad} x$. More generally, this defines a map $\operatorname{ad} : L \to \mathfrak{gl}(L)$ such that $x \mapsto \operatorname{ad} x$. This map is called the **adjoint representation** of L.

As mentioned before, the map $\operatorname{ad} x$ is a derivation of L.

Proposition 0.7. *For $x \in L$, $\operatorname{ad} x \in \operatorname{Der} L$.*

Proof. We simply verify the definition. By the anti-commutativity of the bracket (Proposition 0.1), we may rewrite the Jacobi identity as:

$$[x[ab]] - [a[xb]] - [[xa]b] = 0$$

Hence

$$\operatorname{ad} x([ab]) = [x[ab]] = [a[xb]] + [[xa]b] = [a, \operatorname{ad} x(b)] + [\operatorname{ad} x(a), b]$$

which completes the proof. □

Remark 0.8. *Derivations that can be written as $\operatorname{ad} x$ for some $x \in L$ are called **inner** derivations, and all others are called **outer** derivations.*

Remark 0.9. *Sometimes, when x is an element of K, a subalgebra of L, we then need to specify whether x is acting of K or L to avoid ambiguity. Notation-wise, we shall use $\operatorname{ad}_L x$ and $\operatorname{ad}_K x$ for distinction.*

When a basis of L is chosen, expressing $\operatorname{ad} x$ as a matrix makes computations more convenient:

Example 0.7: Matrix form of $\operatorname{ad} x$

[cf. Exercise 3 and Exercise 11] Let (x, h, y) be an ordered basis for $\mathfrak{sl}(2, F)$ as given in the previous subsection. Performing matrix multiplications, we see that:

$$\operatorname{ad} x(x) = 0, \quad \operatorname{ad} x(h) = -2x, \quad \operatorname{ad} x(y) = h$$

Hence, the matrix representation of $\operatorname{ad} x$ is:

$$\operatorname{ad} x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly, using the same method:

$$\operatorname{ad} y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad \operatorname{ad} h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If we consider the product of the matrix representations of $\operatorname{ad} x$ and $\operatorname{ad} h$, then we have:

$$(\operatorname{ad} x)(\operatorname{ad} h) = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By the calculation above, we already have that $[xy] = h = (0, 1, 0)$, so

$$(\operatorname{ad} x)(\operatorname{ad} h)([xy]) = (4, 0, 0)$$

However,

$$(\operatorname{ad} x)(\operatorname{ad} h)(y) = (\operatorname{ad} x)(\operatorname{ad} h)(x) = 0$$

so $(\operatorname{ad} x)(\operatorname{ad} h)([xy]) \neq [x, (\operatorname{ad} x)(\operatorname{ad} h)(y)] + [(\operatorname{ad} x)(\operatorname{ad} h)(x), y]$ and the product is not a derivation. This verifies the claim in remark 0.7.

Remark 0.10. *The entries in these matrices give the so-called "structure constants", and they determine the Lie algebra completely in an abstract manner, as we shall see in the next subsection.*

Abstract Lie Algebras

In all of our previous investigations, we have taken a more "global" approach in defining Lie algebras, by using familiar operations on known vector spaces that are well-studied. However, we may also take a more abstract approach in the definition as follows.

The first key observation is that using bilinearity, the value of the Lie bracket of any arbitrary vectors is **completely determined** by the value of the bracket of the basis vectors. In other words, if L is any Lie algebra with basis $x_1, ..., x_n$, and suppose $v = a_1x_1 + ... + a_nx_n, w = b_1x_1 + ... + b_nx_n$, then

$$[v, w] = \left[\sum_{i=1}^{n} a_i x_i, \sum_{j=1}^{n} b_j x_j\right] = \sum_{1 \le i \ne j \le n} a_i b_j [x_i x_j]$$

so the entire multiplication table can be recovered simply from the Lie brackets $[x_i x_j]$, or even more specifically, suppose:

$$[x_i x_j] = \sum_{k=1}^{n} a_{ij}^k x_k$$

then the multiplication table is completely determined by a_{ij}^k, called the **structure constants** of the Lie algebra L. In the language of matrices, a_{ij}^k is the (k, j)-entry in the matrix representation of $\operatorname{ad} x_i$.

In general, not any set of scalars $\{a_{ij}^k\}$ can define a Lie algebra structure. The specific relations they have to satisfy are given by the axioms (L1), (L2) and (L3) and are stated in Humphreys. The more significant ones are the ones given by (L2), which shows that $a_{ii}^k = 0$, and anti-commutativity, which implies that $a_{ij}^k = -a_{ji}^k$. If we set all structure constants to 0, then clearly we get the abelian Lie algebra, and in fact, this is the only possible one-dimensional Lie algebra since the definition forces $[ax, bx] = 0$ for any $a, b \in F$. Using this abstract point of view, we can actually do further and classify all two-dimensional Lie algebras

Proposition 0.8 (Classification of two-dimensional Lie algebras)**.** *If L is a two-dimensional Lie algebra with basis x, y, then the only possible bracket operations would be $[xy] = 0$ or $[xy] = x$ up to isomorphism.*

Proof. If $[xy] = 0$, then we yield the abelian Lie algebra. If not, then we claim that there exists a basis x', y' such that $[x'y'] = x'$. We first take x' to be a vector that spans the one-dimensional space of multiples of $[xy]$, and y' to be any vector independent to x'. Suppose $x' = a_1x + b_1y, y' = a_2x + b_2y$, then:

$$[x'y'] = [a_1x + b_1y, a_2x + b_2y] = (a_1b_2 - a_2b_1)[xy]$$

so it is a multiple of $[xy]$, and so must be some multiple of x'. Let $[x'y'] = ax'$, we then replace y' by $a^{-1}y'$, then we finally get $[x'y'] = x'$. □

The natural follow-up question would then be: How can we move from the abstract to the concrete, and construct a more intuitive Lie algebra isomorphic to the ones defined abstract above? The key idea is to use the **adjoint representation**:

Example 0.8: A non-abelian two-dimensional linear Lie algebra

[cf. Exercise 4] We want to find a linear Lie algebra that is isomorphic to the non-abelian two-dimensional Lie algebra defined abstractly in the proof above. To do this, we consider the matrix representations of $\operatorname{ad} x$ and $\operatorname{ad} y$. By a similar computation as in Example 0.7, we have that

$$\operatorname{ad} x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{ad} y = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

These two matrices are linearly independent in $\operatorname{End} \mathbb{R}^2$, and if we consider the subspace spanned by $\operatorname{ad} x$ and $\operatorname{ad} y$, we would see that $[\operatorname{ad} x, \operatorname{ad} y] = \operatorname{ad} x$, so (magically) this subspace is automatically a subalgebra of $\mathfrak{gl}(2, \mathbb{R})$ that is isomorphic to the two-dimensional algebra constructed above. The underlying reason is that the adjoint representation is **injective**, so naturally the Lie algebra will be isomorphic to the image algebra in the general linear algebra. A generalization of this result will be proved in the next section.