Lemmas in Ramsey Theory

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Since mathematical syntax can be contrived by a neural network, I suggest writing in a way that engages human intelligence. Instead of a Platonic view that a theorem is a conjectural belief with a proof, to value the path of creating innovative ideas, we rephrase conjecture as research question and regard theorems as a profound way to summarise nontrivial thought processes. The measure of whether a key idea is useful is illustrated by what can be proven thereby, and yet in proof writing, the central ideas are usually lost in the details. Thus, to appreciate my blog, the reader is invited to be open-minded about a style that may seem non-standard, but which I contend is preferable.

An edge colouring of the complete graph

Let $c : E(K_n) \to \mathbb{N}$ be a colouring of the edges of the complete graph K_n. We suppose the number of colours in the image of c is at most r.

Let $v \in V(K_n)$. There are $n-1$ edges incident to v in K_n, and so if $\sum_{i=1}^{r} k_i = n-1$, then for some i, there are at least k_i edges at v of colour i. By iterating this observation, it is possible to a sizeable clique with a large common neighbourhood in some colour.

The case $r = 2$ with colour imbalance

Suppose $r = 2$, with colours red and blue. If the average red degree is pn, there are at most $\frac{n}{2}$ vertices with red degree $\geq 2pn$. After deleting those, if l is the size of the largest blue clique, there are $\leq 2pnl$ red edges incident to it.

Outside this blue clique, the number of vertices with at least $5pl$ red neighbours in it is at most $\frac{2pnl}{5pl} = 0.4n$, so there are still $0.1n - l$ vertices joined by a red edge to at most a $5p$ proportion of the blue K_l.

There are $\binom{l}{\lfloor 5pl \rfloor}$ possible red neighbourhoods, and so there is a red neighbourhood held in common by at least $(0.1n-l)/\binom{l}{\lfloor 5pl \rfloor}$ other vertices. Thus, we obtain a blue clique of size at least $l(1-5p)$ that is joined by blue edges to as many further vertices. In the literature, such a graph is called a book.

The book lemma

Let X and $Y_1, Y_2, \dots, Y_r$ be disjoint subsets of $V(K_n)$. Also, let p_i be the minimum proportion of colour i neighbours of a vertex $x \in X$ in Y. Fix $\alpha_1, \dots, \alpha_r > 0$ and $\beta = \frac{1}{81}r$, $C = \sqrt{16r^3}$. Then we can select $x \in X$, $Y_i' \subset Y_i$ joined to x in colour i, and $X' \subseteq X$ such that each p_i decreases by at most α_i, and for some $\lambda \geq -1$ and j, p_j increases by $\lambda\alpha_j$. Here, $|Y_i'| = p_i|Y_i|$ and $|X'| \geq \beta \exp(-C\sqrt{\lambda+1})|X|$. This is useful, because if λ is large, this yields a so-called density boost, and if λ is small, x is joined to Y_i' in colour i and to $\frac{|X'|-1}{r}$ vertices of X' in the same colour.

Proving this fact, which I call the book lemma, involves elementary analysis and probability. Equipped with the book lemma, one can deduce that if $n \geq (re^{-\delta})^{rk}$, where $\delta = 2^{-160}r^{-12}$, then there is a complete subgraph of size k in some colour.