Martingales - Chapter 28

Huai Jie (Dante) • 28 Apr 2026

Kolmogorov's Continuity Theorem, Brownian Motion, and Related Results

Kolmogorov's Continuity Criterion

Kolmogorov sought the condition that a stochastic process $X = \{X_t : t \in T\}$ must satisfy in order to have a continuous modification. The key lies in a good bound for $P(|X_t - X_s| \geq \varepsilon)$ when s is close to t; by Markov's inequality this depends on the moments of $|X_t - X_s|$.

A second question: can we measure the "degree of smoothness" of the (continuous) sample paths thus obtained?

Definition 1. *Let $f : [a, b] \to \mathbb{R}$ be continuous. We say that f is* ***locally Hölder continuous of order*** *$\alpha \in (0, 1]$ if there exists a constant C such that*

$$|f(s) - f(t)| \leq C\,|s - t|^{\alpha} \qquad \text{for all } a < s < t < b.$$

α is the ***Hölder exponent*** *and C the* ***Hölder constant****. The set of all such functions is denoted by $C^{\alpha}[a, b]$. α-Hölder continuity implies uniform continuity. For $\alpha = 1$ we obtain Lipschitz continuity. As α increases, the condition becomes stronger (the function is smoother).*

Theorem 2 (Kolmogorov continuity theorem)**.** *Let $X = \{X_t : t \in [0, 1]^d\}$ be a stochastic process on $(\Omega, \mathcal{A}, P)$. Suppose there exist constants $p, \varepsilon > 0$ and $C > 0$ such that for all $s, t \in [0, 1]^d$,*

$$\mathbb{E}\big[|X_t - X_s|^p\big] \leq C\,|t - s|^{d+\varepsilon}.$$

Then X has a continuous modification whose sample paths are locally Hölder continuous of order α for every $\alpha \in \left(0, \frac{\varepsilon}{p}\right)$.

Lévy's Modulus of Continuity for Brownian Motion

Let $\{W_t : t \geq 0\}$ be a standard Brownian motion.

Theorem 3 (Lévy's modulus of continuity, 1937)**.**

$$\limsup_{h \to 0} \frac{\sup_{t \in [0,1-h]} |W(t+h) - W(t)|}{\sqrt{2h \log \frac{1}{h}}} = 1 \quad \textit{almost surely.}$$

Equivalently:

- *If $c > 1$, there exists $h_c > 0$ such that for all $h < h_c$,*

$$|W(t+h) - W(t)| < c\sqrt{2h \log \frac{1}{h}} \qquad a.s.$$

- *If $c < 1$, then almost surely,*

$$|W(t+h) - W(t)| > c\sqrt{2h \log \frac{1}{h}}.$$

Remark 4. The upper bound follows from a standard argument (Theorem 2, not reproduced here); the lower bound requires a more refined argument (see Karatzas & Shreve, pp. 114–116). Theorem 3 specifies the precise modulus of continuity for Brownian motion. Almost sure sample continuity and Hölder continuity of Brownian paths for $\alpha \in (0, \frac{1}{2})$ follow from it.

Scaling and Fractal Properties of Brownian Motion

Many natural sets derived from Brownian sample paths can be regarded as "random fractals". This relies on the scaling invariance property.

Lemma 5 (Scaling invariance)**.** *Let $\{W_t : t \geq 0\}$ be a standard Brownian motion and $c \geq 0$. Then the process*

$$X_t = \frac{1}{c} W_{c^2 t}$$

is also a standard Brownian motion.

Proof. Path continuity and the independence and stationarity of increments are preserved because

$$\frac{1}{c}\big(W_{c^2 t} - W_{c^2 s}\big) \sim N(0, t-s).$$

□

Let $a < 0 < b$ and define the first exit time

$$T(a,b) = \inf\{s > 0 : W_s = a \text{ or } W_s = b\}.$$

Using $X_t = \frac{1}{a} W_{a^2 t}$ and setting $s = a^2 t$, we have $W_t = a$ corresponds to $X_t = 1$, and $W_t = b$ corresponds to $X_t = b/a$. Hence

$$\mathbb{E}[T(a,b)] = a^2\,\mathbb{E}\big[\inf\{t \ge 0 : X_t = 1 \text{ or } X_t = b/a\}\big] = a^2\,\mathbb{E}[T(1, b/a)].$$

In particular, $\mathbb{E}[T(-b,b)] = b^2\,\mathbb{E}[T(-1,1)]$, i.e., a constant multiple of b^2. Moreover,

$$P\big(\{W_t\} \text{ exits at } a\big) = P\big(\{X_t\} \text{ exits at } 1\big),$$

which is a function of the ratio b/a only.

Time Inversion

Define the process $\{X_t : t \ge 0\}$ by

$$X_t = \begin{cases} t\,W(1/t), & t > 0, \\ 0, & t = 0. \end{cases}$$

Then $\{X_t\}$ is also a standard Brownian motion.

Proof. The finite-dimensional distributions are Gaussian with mean zero and covariance $\mathrm{Cov}(X_t, X_s) = t \wedge s$. For $t > 0, h > 0$,

$$\mathrm{Cov}(X_{t+h}, X_t) = (t+h)t\;\mathrm{Cov}\big(W(1/(t+h)), W(1/t)\big) = \cdots,$$

so the f.d.d. agree with those of W. Continuity for $t > 0$ is clear. For $t = 0$, note that the distribution of $\{X_t : t > 0, t \in \mathbb{Q}\}$ is the same as that of a Brownian motion; hence $\lim_{t \to 0} X_t = 0$ almost surely. Because $\mathbb{Q} \cap (0,a)$ is dense in $(0,a)$ and the paths are continuous on $(0,a)$ almost surely, the process has continuous sample paths almost surely. $\square$

Consequences of Scaling and Inversion

The scaling and inversion invariances tie Brownian motion to two important groups of transformations on $[0,\infty)$. These symmetries are extremely useful.

- **Scaling invariance**: if we have one interval of a second of a Brownian path, we can expand it to an interval of seconds of an equally valid Brownian path.
- **Inversion invariance**: the first second of the life of a Brownian path is rich enough to capture the behavior of a Brownian path from the end of the first second until the end of time.

Problems

Problem 1. Show that almost all Brownian sample paths are *not* Hölder continuous of order $\alpha = \frac{1}{2}$ using Lévy's modulus of continuity (Theorem 3).

Problem 2. Let $Z \sim \mathcal{N}(0,1)$.

(a) Show that for all $a > 0$,

$$\frac{a}{a^2+1}\frac{1}{\sqrt{2\pi}}e^{-a^2/2} \;\leq\; P(Z \geq a) \;<\; \frac{1}{a}\frac{1}{\sqrt{2\pi}}e^{-a^2/2}.$$

(Hint: analyze $g(a) := ae^{a^2/2}P(Z \geq a)$.)

(b) Show that $\lim_{a\to\infty} \dfrac{1-\Phi(a)}{a^{-1}\phi(a)} = 1$, where ϕ and Φ are the pdf and cdf of $\mathcal{N}(0,1)$, respectively.

(c) Show that $\Phi(t) - \Phi(s) < t - s$ for all $s < t$.

Problem 3. Let $T(a) = \inf\{s > 0 : W_s = a\}$ for $a \in \mathbb{R}$. Show that

$$T(-a) \overset{d}{=} T(a) \quad \text{and} \quad T(a) \overset{d}{=} a^2 T(1).$$

(Hint: reflection and scaling.)

Problem 4. Fix $a > 0$. Show that $B_t := W_{t+a} - W_a$, $t \in [0,\infty)$, is again a standard Brownian motion.

Problem 5. Show that $P(\sup_{t\geq 0} W_t = \infty) = 1$; hence also $P(\inf_{t\geq 0} W_t = -\infty) = 1$. (Hint: let $Z = \sup_{t\geq 0} W_t$. Show that for any $c > 0$, $\frac{1}{2}Z \overset{d}{=} Z$ so that Z is either 0 or ∞ almost surely. Then show $P(Z = 0) = 0$.)

Problem 6. A process $\{X_t : t \geq 0\}$ with $\mathbb{E}[X_t^2] < \infty$ is called **weakly stationary** if $\mathbb{E}[X_t]$ is constant and $\mathbb{E}[X_s X_t] = g(t-s)$ for $0 \leq s \leq t < \infty$, where $g : \mathbb{R} \to \mathbb{R}$ is even. Which of the following processes are stationary?

(a) $U_t = W_t^2 - t$

(b) $X_t = e^{-at}W_{e^{2at}}$

(c) $Y_t = W_{t+h} - W_t$

(d) $Z_t = W_{e^t}$ for $t > 0$