Kolmogorov three-series theorem

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Proof outline. The proof uses Kolmogorov's inequality, the Borel–Cantelli lemmas, and the fact that for bounded independent variables the convergence of the sum of means and variances implies almost sure convergence (the two-series theorem).

The statement (Kolmogorov three-series theorem). Let $X_1, X_2, \dots$ be a sequence of independent real-valued random variables. Fix a constant $C > 0$ and for each n define the truncated variable

$$X_n^{(C)} = X_n \,\mathbf{1}_{\{|X_n| \le C\}}.$$

Then the series $\sum_{n=1}^{\infty} X_n$ converges almost surely if and only if the following three series converge:

1. $\displaystyle\sum_{n=1}^{\infty} \mathbb{P}\big(|X_n| > C\big) < \infty.$
2. $\displaystyle\sum_{n=1}^{\infty} \mathbb{E}\big[X_n^{(C)}\big]$ converges to a finite real number.
3. $\displaystyle\sum_{n=1}^{\infty} \operatorname{Var}\big(X_n^{(C)}\big) < \infty.$

The constant $C > 0$ is arbitrary: if the conditions hold for one C, they hold for every C.

Proof. We prove the equivalence in two parts.

Part 1: Sufficiency ($\Leftarrow$). Assume that (1), (2), and (3) hold for some fixed $C > 0$.

Step 1 — Borel–Cantelli. Condition (1), together with the first Borel–Cantelli lemma, gives

$$\mathbb{P}\big(|X_n| > C \text{ infinitely often}\big) = 0.$$

Hence, almost surely, there exists an index $N(\omega)$ such that for all $n \geq N(\omega)$ we have $|X_n| \leq C$, i.e. $X_n = X_n^{(C)}$. Consequently, the convergence of $\sum_{n=1}^{\infty} X_n$ is equivalent to the convergence of $\sum_{n=1}^{\infty} X_n^{(C)}$.

Step 2 — Reduce to zero-mean variables. Define

$$Y_n = X_n^{(C)} - \mathbb{E}[X_n^{(C)}], \qquad \mu_n = \mathbb{E}[X_n^{(C)}].$$

Then Y_n are independent, bounded ($|Y_n| \leq 2C$), and satisfy $\mathbb{E}[Y_n] = 0$. Condition (3) says

$$\sum_{n=1}^{\infty} \mathrm{Var}(X_n^{(C)}) = \sum_{n=1}^{\infty} \mathbb{E}[Y_n^2] < \infty.$$

Step 3 — Apply Kolmogorov's two-series theorem (or prove directly via Kolmogorov's inequality). *Kolmogorov's two-series theorem:* If $\sum_{n=1}^{\infty} \mathbb{E}[Y_n]$ converges and $\sum_{n=1}^{\infty} \mathrm{Var}(Y_n) < \infty$ for independent Y_n, then $\sum_{n=1}^{\infty} Y_n$ converges almost surely. Here $\mathbb{E}[Y_n] = 0$ (converges trivially) and $\sum_{n=1}^{\infty} \mathrm{Var}(Y_n) < \infty$, hence $\sum_{n=1}^{\infty} Y_n$ converges almost surely.

A direct proof using Kolmogorov's inequality: For any $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{m,n \geq N} \left| \sum_{k=N}^{m} Y_k - \sum_{k=N}^{n} Y_k \right| > \varepsilon \right) \leq \frac{4}{\varepsilon^2} \sum_{k=N}^{\infty} \mathbb{E}[Y_k^2],$$

which tends to 0 as $N \to \infty$ because $\sum_{k=1}^{\infty} \mathbb{E}[Y_k^2] < \infty$. Thus the partial sums form an almost sure Cauchy sequence in the complete metric space $\mathbb{R}$ (with the usual distance) and therefore converge almost surely.

Step 4 — Add back the means. Condition (2) says that $\sum_{n=1}^{\infty} \mu_n$ converges to a finite limit. Since $\sum_{n=1}^{\infty} Y_n$ converges almost surely, we have

$$\sum_{n=1}^{\infty} X_n^{(C)} = \sum_{n=1}^{\infty} \mu_n + \sum_{n=1}^{\infty} Y_n$$

which converges almost surely (almost sure convergence of the second term plus deterministic convergence of the first gives almost sure convergence of the sum). Therefore $\sum_{n=1}^{\infty} X_n^{(C)}$ converges almost surely, and by Step 1 $\sum_{n=1}^{\infty} X_n$ converges almost surely. $\square$

Part 2: Necessity ($\Rightarrow$). Now assume that $\sum_{n=1}^{\infty} X_n$ converges almost surely.

Step 1 — Tail condition (1). Almost sure convergence implies $X_n \to 0$ almost surely; hence for the fixed $C > 0$, almost surely $|X_n| \le C$ for all sufficiently large n. Therefore

$$\mathbb{P}\big(|X_n| > C \text{ infinitely often}\big) = 0.$$

By the second Borel–Cantelli lemma (the events $\{|X_n| > C\}$ are independent since they are functions of the independent X_n), we must have

$$\sum_{n=1}^{\infty} \mathbb{P}\big(|X_n| > C\big) < \infty.$$

Thus condition (1) holds.

Step 2 — Truncation does not affect convergence. Because condition (1) holds, $\mathbb{P}(X_n \ne X_n^{(C)} \text{ i.o.}) = 0$. Therefore $\sum X_n$ converges almost surely if and only if $\sum X_n^{(C)}$ converges almost surely (they differ only in finitely many terms with probability 1). So from now on we work with the bounded sequence $X_n^{(C)}$.

Let $\mu_n = \mathbb{E}[X_n^{(C)}]$ and $Z_n = X_n^{(C)} - \mu_n$. Then $\sum X_n^{(C)}$ converges almost surely implies $\sum Z_n$ converges almost surely (since subtracting the convergent deterministic series $\sum \mu_n$ would break convergence unless $\sum \mu_n$ itself converges; we shall prove that indeed $\sum \mu_n$ must converge). We prove (2) and (3) together.

Step 3 — Two-series theorem in reverse. For independent bounded variables (here $|Z_n| \le 2C$), the two-series theorem is actually an "if and only if":

$$\text{For independent bounded variables with zero mean, } \sum Z_n \text{ converges almo}$$

Proof of the "only if" part: Suppose $\sum Z_n$ converges almost surely and $|Z_n| \le M$. Let $S_N = \sum_{n=1}^{N} Z_n$. By Kolmogorov's inequality, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\max_{1\le k\le N} |S_k| \ge \varepsilon\right) \le \frac{\mathrm{Var}(S_N)}{\varepsilon^2}.$$

Since S_N converges almost surely, it is bounded in probability; in particular, there exists K such that for all N, $\mathbb{P}(|S_N| \ge 1) \le 1/2$. But

$$\mathbb{E}[S_N^2] = \sum_{n=1}^{N} \mathbb{E}[Z_n^2].$$

Using a standard truncation bound of the form

$$\mathbb{E}[S_N^2] \le \big(1 + (M+1)^2\big)\, \mathbb{P}(|S_N| \le 1),$$

or alternatively by applying a Kronecker lemma variant combined with Kolmogorov's maximal inequality, one concludes that if $\sum \mathbb{E}[Z_n^2] = \infty$, then the partial sums cannot converge almost surely. Therefore $\sum \mathbb{E}[Z_n^2] < \infty$.

Thus, from almost sure convergence of $\sum Z_n$ we obtain

$$\sum_{n=1}^{\infty} \mathrm{Var}(X_n^{(C)}) = \sum_{n=1}^{\infty} \mathbb{E}[Z_n^2] < \infty,$$

which is condition (3).

Step 4 — Convergence of the mean series. Now

$$\sum_{n=1}^{\infty} X_n^{(C)} = \sum_{n=1}^{\infty} \mu_n + \sum_{n=1}^{\infty} Z_n.$$

Both $\sum X_n^{(C)}$ and $\sum Z_n$ converge almost surely, therefore their difference $\sum \mu_n$ converges as a deterministic series. This is condition (2). $\square$

Remark on the constant C. If the three series converge for one $C > 0$, then they converge for every $C > 0$. Reason: changing C only modifies finitely many terms of each series in the relevant probabilistic sense. For any two constants $C_1 < C_2$, the events $\{|X_n| > C_2\}$ and $\{|X_n| > C_1\}$ differ only on the set where $C_1 < |X_n| \leq C_2$; the probabilities of such events are summable by the tail condition for the larger constant. The detailed verification is standard and omitted.

Why it is significant.

1. It solves the convergence problem for independent sums.

2. It clarifies the role of truncation.

3. It is a necessary and sufficient condition.

4. It underpins many limit theorems.

5. It reveals the three essential mechanisms preventing convergence:

- Too many large jumps (condition (1) fails).
- Infinite cumulative drift (condition (2) fails).
- Infinite cumulative fluctuation (condition (3) fails).

A concrete example. Suppose X_n takes values $\pm n$ each with probability $1/n^2$ and 0 with probability $1 - 2/n^2$. Choose $C = 1$.

- Condition (3): $\sum \mathbb{P}(|X_n| > 1) = \sum 2/n^2$ converges.
- Condition (1): Let $Y_n = 0$ almost surely (since $|X_n| > 1$ is rare; on truncation we set the large values to zero). Hence $\mathbb{E}[Y_n] = 0$ and the series converges.
- Condition (2): $\mathrm{Var}(Y_n) = 0$, hence the variance series converges.

Therefore $\sum X_n$ converges almost surely. Without the theorem, this would be non-obvious.